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The Viscosity Iterative Algorithms for the Implicit Double Midpoint Rule of Nonexpansive Mappings in Hilbert Spaces

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Abstract In this paper, we study the viscosity iterative algorithms for the implicit double midpoint rule in real Hilbert space and prove strong convergence of the sequence $\{u_n\}$ to a fixed point of T. As an application we employ our method to obtain an application of it in convex minimization and the solution of Fredholm type of integral equations.

Keywords: *viscosity, implicit double midpoint, hilbert space, Fredohlm integral*

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1. Introduction

Let H be a Hilbert space, $T:H \to H$ be a nonexpansive mapping and $\psi : H \to H$ be a contraction. The viscosity approximation method for nonexpansive mapping in Hilbert spaces was introduced by Moudafi [26] by the following iterative method:

Let K be a closed convex subset of Hilbert space H. Assume that $f:K \to K$ is a contraction and $T:K \to K$ is a nonexpansive mapping. For given $x_0 \in K$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(x_n) \quad \forall n \in \mathbb{N} \text{ and } \alpha_n \in (0,1), \tag{1.1}$$

converges strongly to a fixed point of T under certain conditions, which is a solution to the variational inequality

$$\langle (I-f)\overline{x}, \overline{x}-x\rangle \le 0 \text{ for all } x \in F(T).$$

Moudafi's generalizations are called viscosity approximations. Viscosity approximation methods have been extensively employed in the literature to obtain strong convergence results (cf. [11,21,28,30,34] and references therein).

In 2004, Xu [32] extended the result of Moudafi [26] to uniformly smooth Banach spaces and obtained strong convergence theorem. For related work, see [7,16,37].

In 2006, Marino and Xu [38] introduced the following iterative scheme based on viscosity approximation method, for fixed point problem for a nonexpansive mapping S on H:

$$x_{n+1} = \alpha_n \gamma Q(xn) + (I-\alpha_n \mathbf{B})Sx_n, n=0, \tag{1.2}$$

where Q is a contraction mapping on H with constant $\alpha > 0$, B is a strongly positive self-adjoint bounded linear operator on H with constant $\overline{\gamma} > 0$ and $\overline{\gamma} \in \left(0, \frac{\overline{\gamma}}{\alpha}\right)$. They proved that the sequence $\{x_n\}$ generated by 1.2 converge strongly to the unique solution of the variational inequality

$$\langle (\mathbf{B}-\gamma Q)z, x-z\rangle \ge 0 \quad \forall x \in Fix(S) \tag{1.3}$$

which is the optimality condition for the minimization problem

$$\min_{x \in Fix(S)} \frac{1}{2}\langle \mathbf{B}x, x\rangle - h(x) \tag{1.4}$$

where h is the potential function for γQ.

The implicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations. For related works, we refer to [6,39,40,41,42,43,44] and the references cited therein. For instance, consider the initial value problem for the differential equation $y'(t) = f(y(t))$ with the initial condition $y(0) = y0$, where f is a continuous function from R^d to R^d. The implicit midpoint rule in which generates a sequence $\{y_n\}$ by the following the recurrence relation

$$\frac{1}{h}(y_{n+1}-y_n) = f\left(\frac{y_{n+1}-y_n}{2}\right). \tag{1.5}$$

In 2014, implicit midpoint rule has been extended by Alghamdi et al. [45] to nonexpansive mappings, which generates a sequence $\{x_n\}$ by the following implicit iterative scheme:

$$x_{n+} = \alpha_n x_n + (1-\alpha_n) S\left(\frac{x_n + x_{x+1}}{2}\right), \quad n \geq 0 \qquad (1.6)$$

In 2015, Xu *et al.* [34] extended (1.1) and obtained the following Viscosity implicit mid point method:

Theorem 1.1. *Let H be a Hilbert space, K a closed convex subset of $H, T: K \to K$ a nonexpansive mapping with $S := Fix(T) \neq \emptyset$ and $f: K \to K$ a contraction with coefficient $\alpha \in [0,1)$. For givin $x_1 \in K$ the sequence $\{x_n\}$ generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), n \geq 0 \quad (1.7)$$

satisfying the following conditions:

C1: $\lim_{n\to\infty} \alpha_n = 0$

C2: $\sum_{n=0}^{\infty} \alpha_n = \infty$

C3: $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = 1.$

Then the sequence $\{x_n\}$ converges in norm to a fixed point q of T which is also the unique solution of the variational inequality

$$\langle (I-f)q, x-q \rangle \geq 0, x \in q.$$

Later, Ke and Ma [21] and Cai et al. [11] generalized Theorem 1.1 in the setting of Hilbert space. They proposed the following theorems.

Theorem 1.2. [Ke and Ma] *Let C be a nonempty closed convex subset of the real Hilbert space H. Let $T: C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f: C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T(s_n x_n + (1-s_n) x_{n+1}) \quad (1.8)$$

where $\{x_n\}, \{s_n\} \in (0,1)$ satisfying certain conditions, then the sequence $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T which is also the unique solution of the variational inequality

$$\langle (1-f)x, y-x \rangle \geq 0, \forall y \in F(T).$$

Theorem 1.3. [Ke and Ma] *Let C be a nonempty closed convex subset of the real Hilbert space H. Let $T: C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f: C \to C$ be a contraction with coefficient $\theta \in [0,1)$. Pick any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by.*

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T(s_n x_n + (1-s_n) x_{n+1}) \quad (1.9)$$

where $\{x_n\}, \{\beta_n\}, \{\gamma_n\}, \{s_n\} \in (0,1)$ satisfying certain conditions, then the sequence $\{x_n\}$ converges strongly to a fixed point x^* of the nonexpansive mapping T which is also the unique solution of the variational inequality

$$\langle (1-f)x, y-x \rangle \geq 0 \ \ \forall y \in F(T).$$

Recently, Motivated by Xu et al. [34], Tang and Bao [30] considered the following result:

Theorem 1.4. *Let E be a nonempty closed uniformly convex and 2-uniformly smooth Banach space with dual E^*. Let $A: E \to E^*$ be a L-Lipschitz continuous monotone mapping such that $A^{-1}(0) \neq \emptyset$. For given $x_0 \in E$ the sequence defined by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma(I - \omega_n AJ)\left(\frac{x_n + x_{n+1}}{2}\right)$$

where J is the normalized duality mapping. Suppose that $C_{min} \subset (AJ)^{-1} \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element $x^{\dagger 1} \in (AJ)^{-}$ (0).

Of recently many work has not yet been done for viscosity implicit double midpoint rule (VIDMR). The recent work done for (VIDMR) was done by Shrijana Dhakal and Wutiphol Sintunavarat in 2019 where they defined the sequence $\{x_n\}$ in the following theorem.

Theorem 1.5. Shrijana Dhakal and Wutiphol Sintunavarat [46]. *Let C be a nonempty closed convex subset of a real Hilbert space $H, T: C \to C$ be nonexpansive mapping with $Fix(T) \neq \emptyset$, and $f: C \to C$ be contraction mapping with the contractive constant $\alpha \in [0,1)$. Define a sequence $\{x_n\}$ by the following viscosity method for implicit double midpoint rule (VIDMR) as follows*:

$$x_0 \in C$$

$$x_{n+1} = \alpha_n f\left(\frac{x_n + x_{n+1}}{2}\right) + (I - \alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right) \quad (1.10)$$

where an $\alpha \in (0,1)$ for all $n \in N$ and $\{\alpha_n\}$ satisfies the following conditions:

(i): $\lim_{n\to 0} \alpha_n = 0$

(ii): $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$

Then, the sequence $\{x_n\}$ converges to a fixed point z of T, which is also the unique solution

$$\langle (I-f)z, x-z \rangle \qquad (1.11)$$

Motivated by Xu et al. [34], Tang and Bao [30], Shrijana Dhakal and Wutiphol Sintu-navarat [46] and others, we consider viscosity iterative algorithms for the implicit double midpoint rule for nonexpansion mapping in real Hilbert space. Applications to convex minimization problem and nonlinear Fredholm integral equations are included. The results presented in the paper extend and improve some recent results announced in the current literature.

2. Preliminary Notes

In the sequel, we always assume that H is a real Hilbert space and C is a nonempty, closed, and convex subset of H. The nearest point projection from H onto C, PC, is defined by

$$P_C(x) := \arg\min_{z\in C} \|x-z\|^2, x \in H. \quad (2.1)$$

Namely, $P_C(x)$ is the only point in C that minimizes the objective $\|x-z\|$ over $z \in C$, and $P_C(x)$ is characterized as follows:

$$P_C(x) \in C \text{ and } \langle x - P_C(x), z - P_C(x)\rangle \le 0 \text{ for all } z \in C. \quad (2.2)$$

Mapping $T: C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x-y\|,\ x, y \in C.$$

We use $Fix(T)$ to denote the set of fixed points of T. A mapping $\psi: C \to C$ is said to be contractive if there exists a constant $\rho \in (0,1)$ such

$$\|\psi(x) - \psi(y)\| \le \rho\|x-y\| \quad (2.4)$$

for all $x, y \in C$. In this case, ψ is called $\rho-$contraction.

Lemma 2.1. [29]. *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} = (1-\theta_n)a_n + \sigma_n,\ n \ge 0$$

where $\{\theta_n\}$ and $\{\sigma_n\}$ are real sequences such that

(i) $\lim_{n\to\infty} \theta_n = 0,\ \sum_{n=1}^{\infty} \theta_n = \infty$

(ii) $\lim_{n\to\infty} \frac{\sigma_n}{\theta_n} \le 0,\ \sum_{n=1}^{\infty} \sigma_n < \infty.$

Then the sequence $\{a_n\}$ converges to 0.

3. Main Results

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space $H, T: C \to C$ a nonexpansive mapping with $F(T) \ne \varnothing$, and $\psi: C \to C$ a contraction with coefficient a $\rho \in [0,1)$. Let $\{u_n\}$ be generated by the following viscosity implicit double midpoint rule (VIDMR):*

$$u_{n+1} = \eta_n \psi\left(\frac{u_n + u_{n+1}}{2}\right) + (I-\eta_n)T\left(\frac{u_n + u_{n+1}}{2}\right) \forall n \in \mathbb{N} \quad (3.1)$$

where $\{\eta_n\}$ is a sequence in $(0,1)$ such that:

(A1) $\lim_{n\to 0} \eta_n = 0$

(A2) $\sum_{n=0}^{\infty} \eta_n = \infty$

(A3) $\sum_{n=0}^{\infty} |\eta_{n+1} - \eta_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\eta_{n+1}}{\eta_n} = 1.$

Then $\{u_n\}$ converges strongly to a fixed point q of T, which is also the unique solution of the following variational inequality:

$$\langle (1-\psi)q, u-q\rangle$$

Proof. The proof is in five stages.

Step 1: We prove that $\{x_n\}$ is bounded.

Fixing any $p \in F(T)$, we have

$$\begin{aligned}
&\|u_{n+1} - p\| \\
&\le \left\|\eta_n \psi\left(\frac{u_n + u_{n+1}}{2}\right) + (I-\eta_n)T\left(\frac{u_n + u_{n+1}}{2}\right) - p\right\| \\
&\le \left\|\eta_n \psi\left(\frac{u_n + u_{n+1}}{2}\right) - p\right\| + (I-\eta_n)\left\|T\left(\frac{u_n + u_{n+1}}{2}\right) - p\right\| \\
&\le \frac{\eta_n}{2}\|\psi(u_n) - p\| + \frac{\eta_n}{2}\|\psi(u_{n+1}) - p\| \\
&+ \frac{(I-\eta_n)}{2}\|u_n - p\| + \frac{(I-\eta_n)}{2}\|u_{n+1} - p\| \\
&\le \frac{\eta_n}{2}\|\psi(u_n) - \psi(p)\| + \frac{\eta_n}{2}\|\psi(p) - p\| \\
&+ \frac{\eta_n}{2}\|\psi(u_{n+1}) - \psi(p)\| \\
&+ \frac{\eta_n}{2}\|\psi(p) - p\| \frac{(I-\eta_n)}{2}\|u_n - p\| + \frac{(I-\eta_n)}{2}\|u_{n+1} - p\| \\
&\le \frac{\eta_n\rho}{2}\|u_n - p\| + \frac{\eta_n}{2}\|\psi(p) - p\| + \frac{\eta_n\rho}{2}\|u_{n+1} - p\| \\
&+ \frac{\eta_n}{2}\|\psi(p) - p\| \frac{(I-\eta_n)}{2}\|u_n - p\| + \frac{(I-\eta_n)}{2}\|u_{n+1} - p\| \\
&\le \left[\frac{\eta_n\rho}{2} + \frac{(I-\eta_n)}{2}\right]\|u_n - p\| + \eta_n\|\psi(p) - p\| \\
&+ \left[\frac{\eta_n\rho}{2} + \frac{(I-\eta_n)}{2}\right]\|u_{n+1} - p\|
\end{aligned}$$

It then follows that

$$\left[1 - \frac{\eta_n\rho}{2} + \frac{(I-\eta_n)}{2}\right]\|u_{n+1} - p\| \le \left[\frac{\eta_n\rho}{2} + \frac{(I-\eta_n)}{2}\right]\|u_n - p\| + \eta_n\|\psi(p) - p\|$$

Therefore

$$\begin{aligned}
\|u_{n+1} - p\| &\le \left[\frac{1-\eta_n(1-\rho)}{1+\eta_n(1-\rho)}\right]\|u_n - p\| \\
&+ \frac{2\eta_n}{1+\eta_n(1-\rho)}\|\psi(p) - p\| \\
&= \left[1 - \frac{2\eta_n(1-\rho)}{1+\eta_n(1-\rho)}\right]\|u_n - p\| \\
&+ \frac{2\eta_n(1-\rho)}{1+\eta_n(1-\rho)}\left(\frac{1}{1-\rho}\|\psi(p) - p\|\right).
\end{aligned}$$

Consequently we have

$$\|u_{n+1} - p\| \le \max\left\{\|u_n - p\|, \frac{1}{1-\rho}\|\psi(p) - p\|\right\}.$$

By induction, it is easy to see that

$$\|u_n - p\| \le \max\left\{\|u_0 - p\|, \frac{1}{1-\rho}\|\psi(p) - p\|\right\}.$$

Hence $\{u_n\}$ is bounded for all n.

Step 2: We now show that $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$

$$\begin{aligned}
&\|u_{n+1} - u_n\| \\
&= \left\| \begin{array}{l} \eta_n\psi\left(\frac{u_n + u_{n+1}}{2}\right) + (I - \eta_n)T\left(\frac{u_n + u_{n+1}}{2}\right) \\ -\eta_{n-1}\psi\left(\frac{u_{n-1} + u_n}{2}\right) - (I - \eta_{n-1})T\left(\frac{u_{n-1} + u_n}{2}\right) \end{array} \right\| \\
&= \left\| \begin{array}{l} \eta_n\psi\left(\frac{u_n + u_{n+1}}{2}\right) - \eta_n\psi\left(\frac{u_{n-1} + u_n}{2}\right) \\ +(I - \eta_n)T\left(\frac{u_n + u_{n+1}}{2}\right) \\ -(I - \eta_n)T\left(\frac{u_{n-1} + u_n}{2}\right) + \eta_n\psi\left(\frac{u_{n-1} + u_n}{2}\right) \\ -\eta_{n-1}\psi\left(\frac{u_{n-1} + u_n}{2}\right) + (I - \eta_n)T\left(\frac{u_{n-1} + u_n}{2}\right) \\ -(I - \eta_{n-1})T\left(\frac{u_{n-1} + u_n}{2}\right) \end{array} \right\| \\
&\le \frac{\eta_n\rho}{2}\|u_n - u_{n-1}\| + \frac{\eta_n\rho}{2}\|u_{n+1} - u_n\| \\
&+ \frac{I - \eta_n}{2}\|u_n - u_{n-1}\| + \frac{I - \eta_n}{2}\|u_{n+1} - u_n\| \\
&+(\eta_n - \eta_{n-1})\left\|\psi\left(\frac{u_{n-1} + u_n}{2}\right) - T\left(\frac{u_{n-1} + u_n}{2}\right)\right\| \\
&\le \frac{I - \eta_n(1-\rho)}{2}\|u_n - u_{n-1}\| + \frac{1 - \eta_n(1-\rho)}{2}\|u_{n+1} - u_n\| \\
&\quad + |\eta_n - \eta_{n-1}|\xi
\end{aligned}$$

where $\xi = \sup\left\{\left\|\psi\left(\frac{u_n + u_{n+1}}{2}\right) - T\left(\frac{u_n + u_{n+1}}{2}\right)\right\|\right\}$, then we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &\le \frac{I - \eta_n(1-\rho)}{1 + \eta_n(1-\rho)}\|u_n - u_{n-1}\| \\
&+ |\eta_n - \eta_{n-1}|\frac{2\xi}{1 + \eta_n(1-\rho)}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\le \left[1 - \frac{2\eta_n(1-\rho)}{1 + \eta_n(1-\rho)}\right]\|u_n - u_{n-1}\| + |\eta_n - \eta_{n-1}|\frac{2\xi}{1 + \eta_n(1-\rho)}.
\end{aligned}$$

Hence from lemma 2.1 $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$. This implies that $u_n \to 0$ as $n \to \infty$.

Step 3: We now show that $\lim_{n\to\infty}\|u_n - Tu_n\| = 0$. This follows from the argument below

$$\begin{aligned}
&\|u_n - Tu_n\| \le \|u_n - u_{n+1}\| + \|u_{n+1} - Tu_n\| \\
&\le \|u_n - u_{n+1}\| + \eta_n\left\|\psi\left(\frac{u_n + u_{n+1}}{2}\right) - T\left(\frac{u_n + u_{n+1}}{2}\right)\right\| \\
&+\left\|T\left(\frac{u_n + u_{n+1}}{2}\right) - Tu_n\right\|
\end{aligned}$$

$$\le \|u_n - u_{n+1}\| + \eta_n\xi + \frac{1}{2}\|u_{n+1} - u_n\| \le \frac{3}{2}\|u_n - u_{n+1}\| + \eta_n\xi$$

It now follows that $\lim_{n\to\infty}\|u_n - Tu_n\| = 0.$

Step 4: Again we prove that

$$\limsup_{n\to\infty}\langle\psi(q) - q, u_n - q\rangle \le 0 \qquad (3.2)$$

where $q \in Fix(T)$ is the unique fixed point of the contraction $p_{Fix(T)}\psi$, that is $q = p_{Fix(T)}\psi$. Since the sequence $\{u_n\}$ is bounded, then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to p. Thus

$$\limsup_{n\to\infty}\langle\psi(q) - q, u_n - q\rangle = \lim_{n\to\infty}\langle\psi(q) - q, u_{n_k} - q\rangle \quad (3.3)$$

Since $p \in Fix(T)$, then by 2.2, 3.2 and 3.3, we concludes that

$$\limsup_{n\to\infty}\langle\psi(q) - q, u_n - q\rangle = \langle\psi(q) - q, p - q\rangle \le 0. \qquad (3.4)$$

Step 5: We now prove that $u_n \to q \in Fix(T)$ as $n \to \infty$

$$\begin{aligned}
&\|u_{n+1} - q\|^2 \\
&= \left\|\eta_n\psi\left(\frac{u_n + u_{n+}}{2}\right) + (1 - \eta_n)T\left(\frac{u_n + u_{n+}}{2}\right) - q\right\|^2 \\
&= \eta_n\left\langle\psi\left(\frac{u_n + u_{n+}}{2}\right) - q, u_{n+1} - q\right\rangle \\
&+(1 - \eta_n)\left\langle T\left(\frac{u_n + u_{n+}}{2}\right) - q, u_{n+1} - q\right\rangle \\
&= \frac{\eta_n}{2}\left\langle\psi(u_n) - q, u_{n+1} - q\right\rangle \\
&+ \frac{\eta_n}{2}\left\langle\psi(u_{n+1}) - q, u_{n+1} - q\right\rangle \\
&+(1 - \eta_n)\left\langle T\left(\frac{u_n + u_{n+}}{2}\right) - q, u_{n+1} - q\right\rangle \\
&= \frac{\eta_n}{2}\left\langle\psi(u_n) - \psi(q), u_{n+1} - q\right\rangle \\
&+ \frac{\eta_n}{2}\left\langle\psi(q) - q, u_{n+1} - q\right\rangle \\
&+ \frac{\eta_n}{2}\left\langle\psi(u_{n+1}) - \psi(q), u_{n+1} - q\right\rangle \\
&+ \frac{\eta_n}{2}\left\langle\psi(q) - q, u_{n+1} - q\right\rangle \\
&+(1 - \eta_n)\left\langle T\left(\frac{u_n + u_{n+}}{2}\right) - q, u_{n+1} - q\right\rangle
\end{aligned}$$

$$\leq \frac{\eta_n\rho}{2}\|u_n-q\|\|u_{n+1}-q\|+\eta_n\langle\psi(q)-q,u_{n+1}-q\rangle$$
$$+\frac{\eta_n\rho}{2}\|u_{n+1}-q\|\|u_{n+1}-q\|+\frac{(1-\eta_n)}{2}\|u_n-q\|\|u_{n+1}-q\|$$
$$+\frac{(1-\eta_n)}{2}\|u_{n+1}-q\|\|u_{n+1}-q\|$$

$$\leq \frac{\eta_n\rho}{2}\|u_n-q\|\|u_{n+1}-q\|+\eta_n\langle\psi(q)-q,u_{n+1}-q\rangle$$
$$+\frac{\eta_n\rho}{2}\|u_{n+1}-q\|^2+\frac{(1-\eta_n)}{2}\|u_n-q\|\|u_{n+1}-q\|$$
$$+\frac{(1-\eta_n)}{2}\|u_{n+1}-q\|^2$$
$$\leq \frac{1-\eta_n(1-\rho)}{2}\|u_n-q\|\|u_{n+1}-q\|+\eta_n\langle\psi(q)-q,u_{n+1}-q\rangle$$
$$+\frac{1-\eta_n(1-\rho)}{2}\|u_{n+1}-q\|^2.$$

Thus, we have the following

$$\frac{3+\eta_n(1-\rho)}{4}\|u_{n+1}-q\|^2$$
$$\leq \frac{1-\eta_n(1-\rho)}{4}\|u_n-q\|^2+\eta_n\langle\psi(q)-q,u_{n+1}-q\rangle.$$

$$\|u_{n+1}-q\|^2 \leq \left(\frac{1-\eta_n(1-\rho)}{3+\eta_n(1-\rho)}\right)\|u_n-q\|^2$$
$$+\frac{4\eta_n}{3+\eta_n(1-\rho)}\eta_n\langle\psi(q)-q,u_{n+1}-q\rangle$$
$$\leq \left(1-\frac{2-2\eta_n(1-\rho)}{3+\eta_n(1-\rho)}\right)\|u_n-q\|^2$$
$$+\frac{4\eta_n}{3+\eta_n(1-\rho)}\eta_n\langle\psi(q)-q,u_{n+1}-q\rangle.$$

Therefore from lemma 2.1, we can see that $\theta=\frac{2-2\eta_n(1-\rho)}{3+\eta_n(1-\rho)}$ and

$$\sigma=\frac{4\eta_n}{3+\eta_n(1-\rho)}\eta_n\langle\psi(q)-q,u_{n+1}-q\rangle.$$

Hence we can conclude that $u_n \to q$. This completes the proof.

Theorem 3.2. *Let C be a closed convex subset of a Hilbert space $H, T: C \to C$ a nonexpansive mapping with $F(T) \neq \varnothing$, and $\psi: C \to C$ a contraction with coefficient a $\rho \in [0,1)$. Let ω be a constant. Let $\{u_n\}$ be generated by the following viscosity implicit double midpoint rule (VIDMR):*

$$\mathrm{u}_{n+1}=\eta_n\omega+(I-\eta_n)T\left(\frac{u_n+u_{n+1}}{2}\right)\forall_n \in \mathbb{N}.$$

where $\{\eta_n\}$ is a sequence in $(0,1)$ such that:

(A1) $\lim_{n\to 0}\eta_n=0$

(A2) $\sum_{n=0}^{\infty}\eta_n=\infty$

(A3) $\sum_{n=0}^{\infty}|\eta_{n+1}-\eta_n|<\infty$ *or* $\lim_{n\to\infty}\frac{\eta_{n+1}}{\eta_n}=1$

Then $\{u_n\}$ converges strongly to a fixed point q of T, which is also the unique solution of the following variational inequality:

$$\langle(1-\psi)q,u-q\rangle.$$

Corollary 3.3. *Let C be a closed convex subset of a Hilbert space $H, T: C \to C$ a nonexpansive mapping with $F(T) \neq \varnothing$, and $\psi: C \to C$ a contraction with coefficient a $\rho \in [0,1)$. Let ω be a constant. Let $\{u_n\}$ be generated by the following viscosity implicit double midpoint rule (VIDMR):*

$$u_{n+1}=\eta_n\omega+(I-\eta_n)T\left(\frac{u_n+u_{n+1}}{2}\right)\forall n \in \mathbb{N} \qquad (3.5)$$

where $\{\eta_n\}$ is a sequence in $(0,1)$ such that: Then $\{u_n\}$ converges strongly to a fixed point q of T, which is also the unique solution of the following variational inequality:

$$\langle(1-\psi)q,u-q\rangle.$$

Here we assumed $\psi\left(\frac{u_n+u_{n+1}}{2}\right)=\omega.$

Corollary 3.4. *Let C be a closed convex subset of a Hilbert space $H, T: C \to C$ a nonexpansive mapping with $F(T) \neq \varnothing$, and $\psi: C \to C$ a contraction with coefficient a $\rho \in [0,1)$. Let $\{u_n\}$ be generated by the following viscosity implicit double midpoint rule (VIDMR):*

$$u_{n+1}=\eta_n\psi\left(\frac{u_n+u_{n+1}}{2}\right)+(I-\eta_n)T\left(\frac{u_n+u_{n+1}}{2}\right)\forall n \in \mathbb{N} \qquad (3.6)$$

where $\{\eta_n\}$ is a sequence in $(0,1)$ such that:

(A1): $\lim_{n\to 0}\eta_n=0$

(A2): $\sum_{n=0}^{\infty}\eta_n=\infty$

(A3): $\sum_{n=0}^{\infty}|\eta_{n+1}-\eta_n|<\infty$ *or* $\lim_{n\to\infty}\frac{\eta_{n+1}}{\eta_n}=1.$

Then $\{u_n\}$ converges strongly to a fixed point q of T, which is also the unique solution of the following variational inequality:

$$\langle(1-\psi)q,u-q\rangle.$$

4. Application to Convex Minimization Problems

In this section, we study the problem of finding a minimizer of a convex function ∇ defined from Hilbert space $C \to \mathbb{R}$.

The following basic results are well known.

Remark 4.1. It is well known that if $\nabla : C \to \mathbb{N}$ be a real-valued differentiable convex function and $a \in C$, then the point a is a minimizer of ∇ on C if and only if $d\nabla(a) = 0$.

Definition 4.2. A function $\nabla : C \to C$ is said to be strongly convex if there exists $\alpha > 0$ such that for every $u, v \in C$ with $u \neq v$ and $\lambda \in (0,1)$, the following inequality holds:

$$\nabla(\lambda u + (1-\lambda)v) \leq \lambda \nabla u + (1-\lambda)\nabla v - \alpha \|u - v\|^2. \quad (4.1)$$

Lemma 4.3. *Let C be normed linear space and $\nabla : C \to \mathbb{N}$ a real-valued differentiable convex function. Assume that ∇ is strongly convex. Then the differential map $d\nabla : C \to C$ is strongly monotone, i.e., there exists a positive constant k such that*

$$\langle d\nabla u - d\nabla v, u - v\rangle \geq k \| u - v \|^2 \; \forall u, v \in C. \quad (4.2)$$

We now prove the following theorem.

Theorem 4.4. *Let C be a closed convex subset of a Hilbert space $H, d\nabla : C \to C$ a nonexpansive mapping with $F(T) \neq \varnothing$, and $\psi : C \to C$ a contraction with coefficient a $\rho \in [0,1)$. Let $\{u_n\}$ be generated by the following viscosity implicit double midpoint rule (VIDMR):*

$$\begin{aligned} u_{n+1} &= \eta_n \psi\left(\frac{u_n + u_{n+1}}{2}\right) \\ &\quad + (I - \eta_n) d\nabla\left(\frac{u_n + u_{n+1}}{2}\right) \forall n \in \mathbb{N}. \end{aligned} \quad (4.3)$$

where $\{\eta_n\}$ is a sequence in $(0,1)$ with conditions $A1$, $A2$ *and $A3$, then $\{u_n\}$ converges strongly to a fixed point q of T, which is also the unique solution of the following variational inequality:*

$$\langle (1-\psi)q, u - q\rangle.$$

Proof. Since C is nonempty closed convex, it follows that $d\nabla$ has a unique minimizer a^* characterized by $d\nabla(a^*) = 0$ (Remark 4.1). Finally, from Lemma 4.3 and the fact that the differential map $d\nabla : C \to C$ is contraction with a with a contraction coefficient $\rho \in [0,1)$, then the proof follows from Theorem 3.1.

5. Fredholm Integral Equation

Let $\mathcal{Y} = L^2[0,1]$ be space of square integrable function $u : [0,1] \to \mathbb{R}$ endowed with inner product $\langle u, v\rangle_2 = \int_0^1 u(x)v(x)dx$. Now we discuss the solution of following Fredholm integral equation:

$$\begin{aligned} &u(x) = \phi(x) + \lambda \int_0^1 \tau(x,y)\varphi(x,y,u(y))d\,y \\ &x, y \in [0,1] = V, \end{aligned} \quad (5.1)$$

and suppose that the following conditions hold: where $\varphi : V \times V \times \mathbb{R} \to \mathbb{R}$ and $\phi : V \to \mathbb{R}$. To obtain our claim, we consider the followings assumptions:

(A1) The functions $\varphi : V \times V \times \mathbb{R} \to \mathbb{R}$ $\phi : V \to \mathbb{R}$. are continuous.

(A2) φ is Lipschitz continuous, that is, for all $u, v \in \mathcal{Y}$.

$$|\varphi(x,y,u) - \varphi(x,y,v)| \leq L\,|u(x) - v(x)|, x \in V \quad (5.2)$$

(A3) $\tau : V \times V \to \mathbb{R}$ is continuous for all $(x,y) \in V \times V$, $|\tau(x,y)| \leq c$, where $c > 0$:

(A4) $\lambda c L = 1$ and $\lambda > 0$:

Now, we consider the mapping $T : \mathcal{Y} \to \mathcal{Y}$ defined as

$$(Tu)(x) = \phi(x) + \lambda \int_0^1 \varphi(x,y,u(y))dy, x \in [0,1] = V, \quad (5.3)$$

It is easy to observe that T is a nonexpansive mapping. For this, for every $u, v \in \mathcal{Y}$

$$\begin{aligned} &|Tu(x) - Tv(x)|^2 \\ &= \left| \begin{matrix} \left(\phi(x) + \lambda \int_0^1 \tau(x,y)\varphi(x,y,u(y))d\,y\right) \\ -\left(\phi(x) + \lambda \int_0^1 \tau(x,y)\varphi(x,y,v(y))d\,y\right) \end{matrix} \right|^2 \\ &= \lambda^2 \left| \int_0^1 \tau(x,y)(\varphi(x,y,u(y)) - \varphi(x,y,v(y)))d\,y \right|^2 \\ &\leq \int_0^1 |\tau(x,y)\varphi(x,y,u(y)) - \varphi(x,y,v(y))|^2 \,d\,y \\ &\leq \lambda^2 \int_0^1 |\tau(x,y)|^2 |\varphi(x,y,u(y)) - \varphi(x,y,v(y))|^2 \,dy \\ &= \lambda^2 c^2 L^2 \int_0^1 |u(y) - v(y)|^2 \,dy. \end{aligned}$$

This implies that

$$\|T(u) - T(v)\| \leq \lambda c L \|u - v\| = \|u - v\|$$

and T is a nonexpansive mapping. Define

$$\mathcal{B} = \{u \in \mathcal{Y} : \| u \| \leq r\} \quad (5.4)$$

where r is sufficiently large, then $\mathcal{B}$ is a closed ball of $\mathcal{Y}$ of radius r with center at origin. It can be easily seen that $T(\mathcal{B}) \subseteq \mathcal{B}$. From Theorem in [8], operator T has a fixed point in $\mathcal{B}$ and this fixed point of operator is a solution of nonlinear integral equation 5.1.

Theorem 5.1. *Let $\mathcal{Y} = L^2[0,1]$ be a Hilbert space defined above and $T : \mathcal{Y} \to \mathcal{Y}$ be a operator defined in (5.3). Let $\psi : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. For arbitrary given $u_0 \in C$, define the sequence $\{u_n\}$ as follows*:

$$u_{n+1} = \eta_n \psi\left(\frac{u_n + u_{n+1}}{2}\right) + (I - \eta_n) T\left(\frac{u_n + u_{n+1}}{2}\right) \quad (5.5)$$

where I is an identity operator and the sequences $\{\eta_n\}$ is in the interval $(0,1]$ satisfying the following conditions

(A1) $\lim_{n \to 0} \eta_n = 0$

(A2) $\sum_{n=0}^{\infty} \eta_n = \infty$

(A3) $\sum_{n=0}^{\infty} |\eta_{n+1} - \eta_n| < \infty$ *or* $\lim_{n\to\infty} \frac{\eta_{n+1}}{\eta_n} = 1.$

then the sequence $\{u_n\}$ *converges weakly to the solution of nonlinear integral equation* (5.1) *and the proof is the required conclusion of Theorem* 3.1.

Example 5.2. Consider the following integral equation:

$$u(x) = \left[\sin(\pi x) - \frac{4}{9\pi}\left(1 + \frac{1}{5\pi}\right)x\right] + \int_0^1 \frac{x(5+y)u(y)}{7} dy, x \in [0,1]. \quad (5.6)$$

The above integral equation is a particular case of 5.1 with

$$\phi(x) = \sin(\pi x) - \frac{4}{9\pi}\left(1 + \frac{1}{5\pi}\right)x$$

$$\text{and } \psi(x, y, u) = \frac{x(5+y)u(y)}{7}.$$

For any $u, v \in \mathbb{R}$ and $x, y \in [0,1]$, we have

$$|\psi(x,y,u) - \psi(x,y,v)| = \left|\frac{x(5+y)u(y)}{7} - \frac{x(5+y)v(y)}{7}\right| \quad (5.7)$$

$$\leq \frac{x(5+y)}{7} |u - v| \leq |u - v|. \quad (5.8)$$

It can be easily seen that $\phi : J \to \mathbb{R}$ is a continuous function. Thus, integral equation 5.6 has a solution. It can be seen that $u(x) = \sin(\pi x)$ is a solution of 5.6.

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