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A Study on a Class of Entire Dirichlet Series in *n* - Variables

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Abstract In this paper a class L of entire functions represented by Dirichlet series in n variables has been considered whose coefficients belong to the set of complex numbers C and is further proved to be a Banach Algebra. Also characterization of continuous linear functional is done for the set L.

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Keywords: *Dirichlet series, Banach algebra, topological zero divisor, division algebra, continuous linear functional*

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1. Introduction

Let

$$f(s_1,s_2,\dots,s_n) = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots \sum_{m_n=1}^{\infty} a_{m_1,m_2,\dots,m_n} e^{(\lambda_{1m_1}s_1+\lambda_{2m_2}s_2+\dots+\lambda_{nm_n}s_n)} \quad (1.1)$$

be a n-tuple Dirichlet series where

$$s_j=\sigma_j+it_j,\ j\in\{1,2,\dots,n\}$$

and $a_{m_1,m_2,\dots,m_n}\in\mathbb{C}$. Also

$$0<\lambda_{p_1}<\lambda_{p_2}<\dots<\lambda_{p_k}\to\infty \text{ as } k\to\infty \text{ for } p=1,2,\dots,n.$$

To simplify the form of n-tuple Dirichlet series, we have the following notations

$$s=(s_1,s_2,\dots,s_n)\in\mathbb{C}^n,$$

$$m=(m_1,m_2,\dots,m_n)\in\mathbb{C}^n$$

and

$$\lambda_{n_{m_n}}=(\lambda_{1m_1},\lambda_{2m_2},\dots,\lambda_{nm_n})\in\mathbb{R}^n.$$

$$\lambda_{n_{m_n}}s=\lambda_{1m_1}s_1+\lambda_{2m_2}s_2+\dots+\lambda_{nm_n}s_n$$

$$|\lambda_{n_{m_n}}|=\lambda_{1m_1}+\lambda_{2m_2}+\dots+\lambda_{nm_n}$$

$$|m|=m_1+m_2+\dots+m_n.$$

Thus, the series (1.1) can be written as

$$f(s)=\sum_{m=1}^{\infty}a_m e^{\lambda_{n_{m_n}}s}. \quad (1.2)$$

Janusauskas in [1] showed that if there exists a tuple $p>\overline{0}=(0,0,\dots,0)$ such that

$$\limsup_{|m|\to\infty}\frac{\sum_{k=1}^{\infty}\log m_k}{p\lambda_{n_{m_n}}}=0, \quad (1.3)$$

then the domain of absolute convergence of (1.2) coincides with its domain of convergence. Sarkar in [2] proved that the necessary and sufficient condition for series (1.2) satisfying (1.3) to be entire is that

$$\lim_{|m|\to\infty}\frac{\log|a_m|}{|\lambda_{n_{m_n}}|}=-\infty. \quad (1.4)$$

Consider L as the set of series (1.2) satisfying (1.3) and (1.4) for which

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|a_m|$$

is bounded. Thus there exists a G such that

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|a_m|<G,$$

$$c_1e\log|m|+c_1e+c_2\frac{\log(|m|!)}{|\lambda_{n_{m_n}}|}+\frac{\log|a_m|}{|\lambda_{n_{m_n}}|}<\frac{\log G}{|\lambda_{n_{m_n}}|},$$

$$\frac{\log|a_m|}{|\lambda_{n_{m_n}}|}<-\left\{c_1 e\log|m|+c_1 e+c_2\frac{\log(|m|!)}{|\lambda_{n_{m_n}}|}+\frac{\log G}{|\lambda_{n_{m_n}}|}\right\}.$$

This implies

$$\frac{\log|a_m|}{|\lambda_{n_{m_n}}|}\to-\infty \text{ as } |m|\to\infty.$$

Then every element of L represents an entire function. Define the binary operations in L as

$$f(s)+h(s)=\sum_{m=1}^{\infty}(a_m+b_m)e^{\lambda_{n_{m_n}}s},$$

$$\gamma f(s)=\sum_{m=1}^{\infty}(\gamma a_m)e^{\lambda_{n_{m_n}}s},$$

$$f(s).h(s)=\sum_{m=1}^{\infty}(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}a_m b_m e^{\lambda_{n_{m_n}}s}.$$

The norm in L is defined as

$$\|f\|=\sup_{|m|\geq 1}(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|a_m|. \qquad (1.5)$$

During the last two decades a lot of research has been carried out in the field of Dirichlet series and many important results have been proved where few of them may be found in [3,4]. Kumar and Manocha in [5,6] considered the condition $(\lambda_n)^{c_1(\lambda_n)}e^{\{c_2 n-c_1\}}(\lambda_n)\|a_n\|$ of weighted norm for a Dirichlet series in one variable and established some results on it. Until now a lot work has been done for the Dirichlet series in one variable. The purpose of this paper is to give a broader view to the study of Dirichlet series in n-variables.

2. Main Results

In this section main results have been proved. For the definitions of terms used, refer [7,8].

Theorem 1. *L is a commutative Banach algebra with identity.*

Proof. To prove the theorem we first show that L is complete under the norm defined by (1.5). Define a metric on L as $d(f,g)=\|f-g\|$ and let $\{f_m:m\in M\}$ be a Cauchy sequence in L. For each $m\in M$ let $R_m=\{f_k:k\geq m\}$ be the m-th tail of sequence and t_m be twice the diameter of R_m. Also let B_m be the closed ball centered at f_m of radius $c_m=2t_m$. Then

$$R_m\subseteq B_m.$$

Since the sequence is Cauchy therefore $\lim_{m\to\infty}t_m=0.$ Now let $m\in M$ be arbitrary. Therefore there exists $k>m$ such that $t_k<\frac{1}{2}t_m$.

Suppose $g(s)\in B_k$ then

$$\begin{aligned}d(g,f_m)&\leq d(g,f_k)+d(f_k,f_m)\\&\leq c_k+t_m=2t_k+t_m\\&<2t_m=c_m.\end{aligned}$$

Therefore $g(s)\in B_m$ and hence $B_k\subseteq B_m$.

In the like manner we construct a nested sequence of the closed balls $\{B_m:m\in M\}$. Then from hypothesis it is known that a space is complete if and only if every nested sequence of closed balls whose radii tends to zero has a non empty intersection say f. Let $\{f_{r_1}\}$ be a Cauchy sequence in L where

$$f_{r_1}(s)=\sum_{m=1}^{\infty}a_m^{(r_1)}e^{\lambda_{n_{m_n}}s}.$$

Then for given $\epsilon>0$ we can find $r\geq 1$ such that

$$\|f_{r_1}-f_{r_2}\|<\epsilon\ \forall\ r_1,r_2\geq r$$

that is

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|a_m^{(r_1)}-a_m^{(r_2)}|<\epsilon\ \forall\ r_1,r_2\geq r.$$

Clearly $\{a_m^{(r_1)}\}$ forms a Cauchy sequence in $\mathbb{C}$ for all values of $m\geq 1$.

Hence

$$\lim_{r_1\to\infty}a_m^{(r_1)}=a_m\ \forall\ m\geq 1.$$

Letting $r_2\to\infty$,

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|a_m^{(r_1)}-a_m|<\epsilon\ \forall\ r_1\geq r.$$

Thus $f_{r_1}\to f$ that is $d(f_{r_1},f)\to 0$ as $r_1\to\infty$. Also

$$\begin{aligned}&\sup_{|m|\geq 1}(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|a_m|\\&\leq\sup_{|m|\geq 1}(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|a_m^{(r_1)}-a_m|\\&+\sup_{|m|\geq 1}(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|a_m^{(r_1)}|.\end{aligned}$$

Hence $f(s)=\sum_{m=1}^{\infty}a_m e^{\lambda_{n_{m_n}}s}\in L$. Thus L is complete under the norm defined by (1.4). If $f(s),g(s)\in L$ then

$$\begin{aligned}\|f.g\|&\leq\sup_{|m|\geq 1}(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|a_m|\cdot\\&\cdot\sup_{|m|\geq 1}(|m|e)^{c_1|\lambda_{n_{m_n}}|e}(|m|!)^{c_2}|b_m|\\&=\|f\|.\|g\|.\end{aligned}$$

The identity element in L is

$$e(s)=\sum_{m=1}^{\infty}(|m|e)^{-c_1|\lambda_{n_{m_n}}|e}(|m|!)^{-c_2}e^{\lambda_{n_{m_n}}s}.$$

This completes the proof of the theorem.

Theorem 2. *The function* $f(s)=\sum_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s}$ *is invertible in* L *if and only if*

$$|k_m (|m|e)^{-c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2}|$$

is a bounded sequence where k_m *is inverse of* a_m.

Proof. Let $f(s)\in L$ be invertible and

$$h(s)=\sum_{m=1}^{\infty} b_m e^{\lambda_{n_{m_n}} s}$$

be its inverse. Then $f(s).h(s)=e(s).$ Therefore

$$\sum_{m=1}^{\infty} (|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} a_m b_m e^{\lambda_{n_{m_n}} s}$$
$$=\sum_{m=1}^{\infty} (|m|e)^{-c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2} e^{\lambda_{n_{m_n}} s}$$

which implies

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} a_m b_m$$
$$=(|m|e)^{-c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2}$$

This further implies

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} b_m$$
$$=(|m|e)^{-c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2} a_m^{-1}.$$

Equivalently

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} |b_m|$$
$$=|k_m (|m|e)^{-c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2}|$$

and is thus a bounded sequence since $h(s)\in L.$ Conversely suppose

$$|k_m (|m|e)^{-c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2}|$$

be a bounded sequence. Define $h(s)$ such that

$$h(s)=\sum_{m=1}^{\infty} (|m|e)^{-2c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-2c_2} a_m^{-1} e^{\lambda_{n_{m_n}} s}.$$

Further

$$f(s).h(s)=\sum_{m=1}^{\infty} (|m|e)^{-c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2} e^{\lambda_{n_{m_n}} s}=e(s).$$

Hence the theorem.

Theorem 3. *A necessary and a sufficient condition that an element* $f(s)\in L$ *to be a topological zero divisor is that*

$$\lim_{m\to\infty} (|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} |a_m|=0.$$

Proof. Let the given condition holds. Construct a sequence $\{h_m\}$ such that

$$h_m=(|m|e)^{-c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2} e^{\lambda_{n_{m_n}} s}.$$

Thus, for all m, $h_m(s)\in L$ and $\| h_m\|=1.$ Now

$$f(s).h_m(s) = h_m(s).f(s)=\sum_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s}.$$

Therefore

$$\| f.h_m\|=\| h_m.f\|=(|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} |a_m|.$$

As $m\to\infty$,

$$\| f.h_m\|=\| h_m.f\|\to 0.$$

Thus $f(s)$ is a topological zero divisor.

Conversely, suppose the given condition is not true that is

$$\lim_{m\to\infty} (|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} |a_m|=\vartheta>0.$$

Then, given ζ with $0<\zeta<\vartheta$ we can find integers $m_0\ge 1$ such that for all $m\ge m_0$,

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} |a_m|\ge \vartheta-\zeta$$

hold true. Also since $f(s)$ is a topological zero divisor, there exists a sequence $\{h_t\}$ of elements in L with unit norm such that for all $t\ge 1$ one has

$$\sum_{t=1}^{\infty} (|t|e)^{c_1|\lambda_{n_{t_n}}|^e} (|t|!)^{c_2} |b_t|=1$$

where

$$h_t(s)=\sum_{t=1}^{\infty} b_t e^{\lambda_{n_{m_n}} s}.$$

Next, for ϵ such that $0<\epsilon<1$ there exist integers M_t and subsequences $\{m_t\}$ of sequence of indices $\{m\}$ such that

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} |b_{m_t}|>1-\epsilon$$

for all $m=m_i\ge M_t$. This implies

$$(|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} \{(|m|e)^{c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} |a_m.b_{m_t}|\}$$
$$>p>0$$

for all $m_i\ge M_t$. Therefore

$$\| f.h_t\| \nrightarrow 0.$$

which is a contradiction. Hence the theorem.

Theorem 4. L *is not a Division Algebra.*

Proof. Let

$$\mathrm{S}\, k(s)=\sum_{m=1}^{\infty} m^{-1} (|m|e)^{-c_1|\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2} e^{\lambda_{n_{m_n}} s}.$$

Clearly $k(s) \in L$ and does not possess an inverse in L. Let if possible

$$w(s) = \sum_{m=1}^{\infty} w_m e^{\lambda_{n_{m_n}} s}$$

be its inverse. Hence $k(s).w(s) = e(s)$. This implies

$$w_m = me_m (|m|e)^{-c_1 |\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2}$$

which does not belong to L. This completes the proof of the theorem.

Theorem 5. *Every continuous linear functional $\phi : L \to \mathbb{C}$ is of the form*

$$\phi(f) = \sum_{m=1}^{\infty} a_m z_m (|m|e)^{c_1 |\lambda_{n_{m_n}}|^e} (|m|!)^{c_2}$$

where

$$f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s}$$

and $\{z_m\}$ is a bounded sequence in $\mathbb{C}$.

Proof. Let us first assume that $\phi : L \to \mathbb{C}$ be a continuous linear functional. Since ϕ is continuous,

$$\phi(f) = \phi(\lim_{M \to \infty} f^{(M)})$$

where

$$f^{(M)}(s) = \sum_{m=1}^{M} a_m e^{\lambda_{n_{m_n}} s}.$$

Let us define a sequence $\{f_m\} \subseteq L$ as

$$f_m = (|m|e)^{-c_1 |\lambda_{n_{m_n}}|^e} (|m|!)^{-c_2} e^{\lambda_{n_{m_n}} s}.$$

Therefore,

$$\phi(f) = \phi\left(\lim_{M \to \infty} \sum_{m=1}^{M} a_m (|m|e)^{c_1 |\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} f_m \right)$$

$$= \lim_{M \to \infty} \sum_{m=1}^{M} a_m (|m|e)^{c_1 |\lambda_{n_{m_n}}|^e} (|m|!)^{c_2} \phi(f_m).$$

Since ϕ is a linear functional therefore

$$\phi(f_m) = z_m.$$

This implies

$$\phi(f) = \lim_{M \to \infty} \sum_{m=1}^{M} a_m z_m (|m|e)^{c_1 |\lambda_{n_{m_n}}|^e} (|m|!)^{c_2}.$$

We now show that $\{z_m\}$ is a bounded sequence in $\mathbb{C}$,

$$|z_m| = |\phi(f_m)| \le G \| f_m \|$$

and $\| f_m \| = 1$ which further implies

$$|z_m| \le G.$$

Thus $\{z_m\}$ is a bounded sequence in $\mathbb{C}$. This proves the theorem.

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