*American Journal of Mathematical Analysis, 2019, Vol. 7, No. 1, 1-10*
Available online at http://pubs.sciepub.com/ajma/7/1/1
Published by Science and Education Publishing
DOI:10.12691/ajma-7-1-1

# A Quaternionic Potential Conception with Applying to 3D Potential Fields

**Michael Parfenov***

Bashkortostan Branch of Russian Academy of Engineering, Ufa, Russia
*Corresponding author: parfenov.48@bk.ru

*Received April 03, 2019; Revised May 17, 2019; Accepted May 28, 2019*

**Abstract** By analogy with complex analysis, any quaternionic holomorphic function, satisfying the earlier presented quaternionic generalization of Cauchy-Riemann's equations (the left- and right conditions of holomorphy together), is defined as a quaternionic potential. It can be constructed by simple replacing a complex variable as single whole by a quaternionic one in an expression for complex potential. The performed "transfer" of the complex potential conception to the quaternionic area is based on the earlier proved similarity of formulas for quaternionic and complex differentiation of holomorphic functions. A 3D model of a potential field, corresponding its quaternionic potential, can be obtained as a result of calculations made in the quaternionic space and the subsequent final transition to 3D space. The quaternionic generalizations of Laplace's equations, harmonic functions and conditions of antiholomorphy are presented. It is shown that the equations of the quaternionic antiholomorphy unite differential vector operations just as the complex ones. The example of applying to the 3D fluid flow modeling is considered in detail.

***Keywords:*** *quaternionic analysis and holomorphic functions, quaternionic Cauchy-Riemann's equations, quaternionic potential, quaternionic Laplace's equations, harmonic functions, 3D models of potential fields*

**Cite This Article:** Michael Parfenov, "A Quaternionic Potential Conception with Applying to 3D Potential Fields." *American Journal of Mathematical Analysis*, vol. 7, no. 1 (2019): 1-10. doi: 10.12691/ajma-7-1-1.

## 1. Introduction

The theory of complex potential plays an important role in the complex analysis when describing the 2-dimensional stationary vector fields (fluid flows, electrostatic fields et al). At that each complex holomorphic ($\mathbb{C}$-holomorphic) function can be considered as the corresponding complex potential, representing some potential (irrotational) vector field and vice versa [1,2]. It is reasonable to ask, whether there is some hypercomplex generalization to describe 3-dimensional stationary potential fields.

*The goal of this work is to show that such a generalization is possible within the framework of the so-called essentially adequate (EA-) quaternionic differentiation theory* [6,7,8].

We first need to recall shortly the conception of complex potential and related notions that are to be generalized to the quaternion case in this article.

Let a continuous $\mathbb{C}$-holomorphic function

$$f(\xi) = u(x,y) + iv(x,y) \tag{1.1}$$

( $\xi = x + iy$ being a complex variable) denote a complex potential, where, by analogy with *a fluid flow*, the function $u(x,y)$ is said to be [1] a *(scalar) velocity potential (or just a potential* when applying to other fields) and the function $v(x,y)$ is said to be a *stream function*. These functions satisfy Cauchy-Riemann's equations of $\mathbb{C}$-holomorphy:

$$\partial_x u(x,y) = \partial_y v(x,y),\ \partial_y u(x,y) = -\partial_x v(x,y) \tag{1.2}$$

where $\partial_x$ and $\partial_y$ denote the first-order derivatives with respect to $x$ and $y$, respectively.

The flow velocity vector

$$F(\xi) = F_1(x,y) + iF_2(x,y) \tag{1.3}$$

is associated with complex potential (1.1) as follows:

$$F(\xi) = \overline{f'(\xi)}, \tag{1.4}$$

where the first derivative of $f(\xi)$ with respect to $\xi$ is denoted by $f'(\xi)$, and the overbar denotes the complex conjugation. We will also use in the sequel the overbar for the designation of the quaternionic conjugation.

The curves

$$u(x,y) = k_1, \tag{1.5}$$

where $k_1$ are real constants, are called *equipotentials.*

The curves

$$v(x,y) = k_2, \tag{1.6}$$

where $k_2$ are real constants, are called the *streamlines* and describe the paths of the fluid particles. The tangent vectors to streamlines coincide with the flow velocity vectors $F(\xi)$ at each point of a flow [1].

The potential nature of the vector field $F(\xi)$ is defined by the following relation:

$$F(\xi) = \nabla u(x, y), \quad (1.7)$$

where $\nabla$ is the gradient operator [1,2,3,4]

$$\nabla = \partial_x + i\partial_y.$$

As is known, the complex conjugate of a $\mathbb{C}$-holomorphic function gives a so-called $\mathbb{C}$-antiholomorphic function [5]. The conditions of $\mathbb{C}$-antiholomorphy can be obtained by direct replacing a $\mathbb{C}$-holomorphic function $f(\xi)$ by its complex conjugate $\overline{f(\xi)} = u(x, y) - iv(x, y)$, i. e. by replacing $u \to u$ and $v \to -v$ in Cauchy-Riemann's equations (1.2), whence follow the conditions of $\mathbb{C}$-antiholomorphy:

$$\partial_x u(x, y) = -\partial_y v(x, y), \partial_y u(x, y) = \partial_x v(x, y). \quad (1.8)$$

Since $f'(\xi)$ is $\mathbb{C}$-holomorphic, the velocity vector $F(\xi)$, according to (1.4), is a $\mathbb{C}$-antiholomorphic function. Using the designations of its constituents in accordance with (1.3), we can rewrite the system of equations (1.8) as follows:

$$\partial_x F_1 = -\partial_y F_2, \partial_y F_1 = \partial_x F_2. \quad (1.9)$$

They represent the differential properties of planar steady state fluid flows in some domains occupied by the fluid [2]:

$$div\, F(\xi) = (\partial_x F_1 + \partial_y F_2) = 0, \quad (1.10a)$$

$$curl\, F(\xi) = (\partial_x F_2 - \partial_y F_1) = 0, \quad (1.10b)$$

where $div\, F(\xi)$ and $curl\, F(\xi)$ denote the divergence and the vorticity of the planar vector field $F(\xi)$. As noted in [2], an "advantage of the complex notation is that it affords a means by which one may condense several vector operations into just one". We will show that the similar "condensation" of vector operations (equations) into one system of equations will be retained in the quaternionic generalization of equations (1.9).

Differentiating Cauchy-Riemann's equations (1.2) with respect to $x$ and $y$, and then combining the results, one can obtain Laplace's equations [1,2,5] for components $u(x, y)$ and $v(x, y)$ and introduce the notion of harmonic functions in the complex plane. We will use this approach to obtain the quaternionic generalization of Laplace's equations and harmonic functions.

As in [6,7,8], we further use for quaternionic variables and functions the following notation. The independent quaternionic variable is denoted by

$$p = x + yi + zj + uk = a + bj \in \mathbb{H}, \quad (1.11)$$

where $x, y, z, u$ are the real variables, and

$$a = x + yi, b = z + ui \quad (1.12)$$

are the complex variables; $i, j, k$ denote the quaternionic basis vectors and $\mathbb{H}$ denotes the quaternion space.

The quaternionic functions are further denoted by

$$\psi(p) = \psi_1 + \psi_2 i + \psi_3 j + \psi_4 k = \phi_1(a,b) + \phi_2(a,b) j \in \mathbb{H}, \quad (1.13)$$

where $\psi_1$, $\psi_2$, $\psi_3$, $\psi_4$ are real-valued functions of real variables $x, y, z, u$ (so-called $\mathbb{R}^4$ –representation [3]) and

$$\phi_1(a, b) = \psi_1(x, y, z, u) + \psi_2(x, y, z, u)i, \quad (1.14)$$

$$\phi_2(a, b) = \psi_3(x, y, z, u) + \psi_4(x, y, z, u)i \quad (1.15)$$

are complex-valued functions of complex variables $a, b$ and their conjugate $\bar{a}, \bar{b}$ (so-called $\mathbb{C}^2$ –representation [3]).

In accordance with [6,7,8] we generalize $\mathbb{C}$-holomorphic functions and Cauchy-Riemann's equations (1.2) to the quaternion case as follows.

**Definition 1.1.** *We assume that the constituents $\phi_1(a, b)$ and $\phi_2(a, b)$ of some quaternionic function $\psi(p) = \psi(a, b) = \phi_1 + \phi_2 j$ possess continuous first-order partial derivatives with respect to $a$, $\bar{a}$, $b$, and $\bar{b}$ in some open connected neighborhood $G_4 \subset \mathbb{H}$ of a point $p \in \mathbb{H}$. Then a function $\psi(p)$ is said to be $\mathbb{H}$-holomorphic and denoted by $\psi_H(p)$ at $p \in G_4$ if and only if the functions $\phi_1(a, b)$ and $\phi_2(a, b)$ satisfy in $G_4$ the following generalized Cauchy-Riemann's equations*:

$$\begin{cases} 1)\ (\partial_a \phi_1| = (\partial_{\bar{b}} \bar{\phi}_2|, & 2)\ (\partial_a \phi_2| = -(\partial_{\bar{b}} \bar{\phi}_1|, \\ 3)\ (\partial_a \phi_1| = (\partial_b \phi_2|, & 4)\ (\partial_{\bar{a}} \phi_2| = -(\partial_{\bar{b}} \phi_1|, \end{cases} \quad (1.16)$$

*where $\partial_i$, $i = a, \bar{a}, b, \bar{b}$, denotes the partial derivative with respect to $i$.*

The brackets $(..|$ with the closing vertical bar indicate that the transition $a = \bar{a} = x$ has been already performed in expressions enclosed in brackets.

Thus, the $\mathbb{H}$-holomorphy conditions are defined so that during the check of the quaternionic holomorphy of any quaternionic function we have to do [6,7,8] the transition $a = \bar{a} = x$ in already computed expressions for the partial derivatives of the functions $\phi_1$ and $\phi_2$ in order to use them in equations (1.16).

This does not mean that we deal with triplets in general, since the transition $a = \bar{a} = x$ cannot be initially done for quaternionic variables and functions [6]. Any quaternionic function remains the same 4-dimensional quaternionic function regardless of whether we check its holomorphy or not. This transition is needed to check the holomorphy of a quaternionic function within the framework of the EA-theory of quaternionic differentiation. However, since this transition is equivalent to the transition to 3-dimensional (3D) space, we will use it to build 3D models of potential fields represented by 4D quaternionic potentials.

Since the quaternionic computations are impossible after such a transition, we further follow the scheme: first doing all needed quaternionic computations with $\mathbb{H}$-holomorphic functions and then performing the transition $a = \bar{a} = x$ to 3D space [6].

According to [6,7], the basic expression for the *full* first-order quaternionic derivative, *uniting the left and right derivatives,* is represented in the Cayley–Dickson doubling form as follows:

$$\psi'_H(p) = (\partial_a \phi_1 + \partial_{\bar{a}} \phi_1) + (\partial_a \phi_2 + \partial_{\bar{a}} \phi_2)j. \quad (1.17)$$

As shown in [6], all expressions for $\mathbb{H}$-holomorphic functions and their derivatives are similar to expressions for the corresponding complex counterparts if $\mathbb{H}$-holomorphic functions are constructed from $\mathbb{C}$-holomorphic functions by replacing a complex variable $\xi$ (as a single whole) by a quaternionic variable $p$. For example, if the first derivative of the $\mathbb{C}$-holomorphic function $\psi_C(\xi) = \xi^n$ is $n\xi^{n-1}$, then the first derivative (1.17) of the $\mathbb{H}$-holomorphic function $\psi_H(p) = p^n$ is $np^{n-1}$.

This is associated with an important property of $\mathbb{H}$-holomorphic functions proved in [8], viz.: generally non-commutative quaternionic multiplication behaves as commutative in the case of the multiplication of $\mathbb{H}$-holomorphic functions defined by equations (1.16).

## 2. Quaternionic Potential and Modeling of 3D Potential Flows

Throughout this article we assume that $\mathbb{H}$ -holomorphic functions and their derivatives are defined in the simply-connected open domains $G_4 \subset \mathbb{H}$. By $(G_4| \subset \mathbb{R}^3$ we denote domains, which result from domains $G_4$ when performing the transition $a = \overline{a} = x$ (in $\mathbb{C}^2$ –representation) or $y = 0$ (in $\mathbb{R}^4$ –representation). In this paper we consider the fluid dynamics, however all results can be applied to potential fields of any nature.

### 2.1. The Quaternionic Potential Conception

According to definition (1.1) of a complex potential as a $\mathbb{C}$- holomorphic function, we introduce the quaternionic generalization of this notion as follows:

**Definition 2.1.** *When considering potential flows (fields) in space we call a $\mathbb{H}$-holomorphic function*

$$\psi_H(p) = \phi_1(a,b) + \phi_2(a,b)j \tag{2.1}$$

*a quaternionic potential, where we refer to the functions $\phi_1(a,b)$ and $\phi_2(a,b)$ as to a potential function (or just a potential) and a stream function, respectively.*

According to the above-mentioned similarity of formulas for quaternionic and complex differentiation, we introduce the following

**Definition 2.2.** *We define the quaternionic generalization of a flow velocity vector (a field vector) as follows:*

$$F(p) = \overline{\psi_H'(p)}, \tag{2.2}$$

*where $\psi_H'(p)$ is a first derivative of a $\mathbb{H}$-holomorphic function in $G_4$ and the overbar denotes the quaternionic conjugation.*

Given (1.17), the expression (2.2) for the generalized flow velocity vector becomes

$$(p) = F_1(a,b) + F_2(a,b)j, \tag{2.3}$$

where

$$F_1(a,b) = \overline{(\partial_a\phi_1 + \partial_{\overline{a}}\phi_1)} = \partial_{\overline{a}}\overline{\phi_1} + \partial_a\overline{\phi_1}, \tag{2.4}$$

$$F_2(a,b) = -(\partial_a\phi_2 + \partial_{\overline{a}}\phi_2). \tag{2.5}$$

It was shown in [6,7] that the system of equations:

$$\begin{aligned} &1)\ \partial_b\phi_2 = \partial_{\overline{b}}\overline{\phi}_2, \quad 2)\ \partial_{\overline{a}}\phi_1 = \partial_a\overline{\phi}_1, \\ &3)\ \partial_a\phi_2 = -\partial_{\overline{b}}\phi_1, \quad 4)\ \partial_{\overline{a}}\phi_2 = -\partial_{\overline{b}}\overline{\phi}_1 \end{aligned} \tag{2.6}$$

holds true (unlike (1.16) as early as before the transition $a = \overline{a} = x$) in an open connected neighborhood $G_4 \subset \mathbb{H}$, if a quaternionic function is $\mathbb{H}$-holomorphic in $G_4$. Given (2.6), the expressions (2.4) and (2.5) become

$$F_1(a,b) = \partial_{\overline{a}}\overline{\phi_1} + \partial_{\overline{a}}\phi_1,$$

$$F_2(a,b) = \partial_{\overline{b}}\phi_1 + \partial_{\overline{b}}\overline{\phi_1}.$$

Thus we have

$$F(p) = (\partial_{\overline{a}}\phi_1 + \partial_{\overline{a}}\overline{\phi_1}) + (\partial_{\overline{b}}\phi_1 + \partial_{\overline{b}}\overline{\phi_1})j. \tag{2.7}$$

This expression represents the quaternionic generalization

$$F(p) = \nabla\phi_1(a,b) \tag{2.8}$$

of the complex expression (1.7). To explain the reason for such a conclusion we write (2.7) in $\mathbb{R}^4$ –representation, using (1.14) and so-called Wirtinger operators [5]:

$$\partial_a = \frac{1}{2}(\partial_x - i\partial_y), \partial_{\overline{a}} = \frac{1}{2}(\partial_x + i\partial_y) \tag{2.9}$$

$$\partial_b = \frac{1}{2}(\partial_z - i\partial_u), \partial_{\overline{b}} = \frac{1}{2}(\partial_z + i\partial_u). \tag{2.10}$$

After some calculation we have

$$F(p) = \partial_x\psi_1 + \partial_y\psi_1 i + \partial_z\psi_1 j + \partial_u\psi_1 k = \nabla\psi_1, \tag{2.11}$$

where $\nabla = \partial_x + \partial_y i + \partial_z j + \partial_u k$ is the gradient operator in the quaternionic space.

Thus, the introduced expression (2.8) for the gradient of the potential function $\phi_1(a,b)$ is equivalent to the gradient of the scalar function $\psi_1(x,y,z,u)$, which is the real part of the expression (1.13) for the quaternionic potential just as the velocity potential $u(x,y)$ is the real part of the expression (1.1) for the complex potential. We see that, formally assuming the definition $\nabla\phi_1(a,b) = \overline{\psi_H'(p)}$, we deals in fact with the gradient $\nabla\psi_1$ of the scalar function $\psi_1$ just like we deals with the gradient of the scalar function in 3D vector calculus.

Now we can perform the transition to 3D space for the introduced quaternionic notions. As shown in [6,7], the equation $(\partial_a\phi_2| = (\partial_{\overline{a}}\phi_2|$ holds true for $\mathbb{H}$-holomorphic functions in $G_4$, whence, by using equations (1.16-2) and (1.16-4), we get

$$(\phi_1| = (\overline{\phi_1}|.$$

Combining this with (1.14), we see that the identity

$$(\psi_2(x,y,z,u)| = 0 \tag{2.12}$$

is always valid for $\mathbb{H}$-holomorphic functions.

Taking into account (2.12), we write the expression (1.13) for the quaternionic potential after transition $y = 0$ (corresponds to $a = \overline{a} = x$) as follows:

$$\begin{aligned}(\psi_H(p)| = &(\psi_1(x,y,z,u)| \\ &+ (\psi_3(x,y,z,u)|\,j + (\psi_4(x,y,z,u)|\,k,\end{aligned} \tag{2.13}$$

*The basis vectors $1, j, k$ represent the basis vectors of 3D space, following from the quaternionic representation.* Correspondingly, the radius vector (1.11) becomes

$$(p| = x + zj + uk.$$

Since after transition $y = 0$ to 3D space all considered expressions are represented in the basis $1, j, k$ (the basis vector $i$ formally vanishes), it is only natural to define the symbolic differential vector-operator $\nabla$ in 3D space also in this basis:

$$\nabla_3 = \partial_x + \partial_z j + \partial_u k. \tag{2.14}$$

To simplify the writing of final results in 3D space we introduce the following notation:

$$\varphi_1(x,z,u) = (\psi_1(x,y,z,u)|,$$
$$\varphi_3(x,z,u) = (\psi_3(x,y,z,u)|, \quad (2.15)$$
$$\varphi_4(x,z,u) = (\psi_4(x,y,z,u)|,$$

for components of the quaternionic potential after transition $y = 0$. We can rewrite (2.13) as follows:

$$(\psi_H(p)| = \varphi_1(x,z,u) + \varphi_3(x,z,u)j + \varphi_4(x,z,u)k,$$

where $\varphi_1(x,z,u)$ can be defined as a scalar potential and $\varphi_3(x,z,u)j + \varphi_4(x,z,u)k$ can be regarded as a stream function in 3D Cartesian space with the basis $1, j, k$.

Performing the transition $y = 0$ in (2.2) and (2.11), we get the following expression in 3D space:

$$(F(p)| = (\overline{\psi'_H(p)}| = (\nabla\psi_1|$$
$$= (\partial_x\psi_1 + \partial_y\psi_1 i + \partial_z\psi_1 j + \partial_u\psi_1 k| \quad (2.16)$$
$$= (\partial_x\psi_1| + (\partial_y\psi_1|i + (\partial_z\psi_1|j + (\partial_u\psi_1|k.$$

We can write expressions for the derivative $\psi'_H(p)$ in $\mathbb{R}^4$ –representation as follows:

$$\psi'_H(p) = \psi'_1 + \psi'_2 i + \psi'_3 j + \psi'_4 k,$$

where $\psi'_1, \psi'_2, \psi'_3, \psi'_4$ are real functions.

The quaternionic conjugation of this expression gives

$$\overline{\psi'_H(p)} = \psi'_1 - \psi'_2 i - \psi'_3 j - \psi'_4 k.$$

After transition $y = 0$ we have

$$(F(p)| = (\overline{\psi'_H(p)}| = (\psi'_1| - (\psi'_2|i - (\psi'_3|j - (\psi'_4|k \quad (2.17)$$

Since a derivative $\psi'_H(p)$ of a $\mathbb{H}$-holomorphic function $\psi_H(p)$ is also $\mathbb{H}$-holomorphic [6,7,8], we get, analogously to (2.12), the following identity:

$$(\psi'_2| = 0. \quad (2.18)$$

Comparing expressions (2.16), (2.17) and taking into account (2.18), we see that

$$(\partial_y\psi_1| = -(\psi'_2| = 0.$$

Then (2.16) becomes

$$(F(p)| = (\overline{\psi'_H(p)}| = (\nabla\psi_1|$$
$$= (\partial_x\psi_1| + (\partial_z\psi_1|j + (\partial_u\psi_1|k.$$

The transition to 3D space is always being carried out by using the variable $y$. However, since differentiating with respect to $y$ in the last expression is not being performed, it is possible, given (2.14) and (2.15), to rewrite this expression as follows:

$$(F(p)| = (\overline{\psi'_H(p)}| = (\nabla\psi_1|$$
$$= \partial_x(\psi_1| + \partial_z(\psi_1|j + \partial_u(\psi_1|k \quad (2.19)$$
$$= \partial_x\varphi_1 + \partial_z\varphi_1 j + \partial_u\varphi_1 k = \nabla_3\varphi_1.$$

We see that the transition from the quaternionic space to 3D one gives the usual 3D expression for the gradient of a scalar function from vector analysis, but in the basis $1, j, k$.

*Thus, the quaternionic potential conception leads to the description of a potential field in 3D space with the basis* $1, j, k$.

The operator $\nabla_3$ can be directly applied to the function $\varphi_1$ obtained from the quaternionic potential after transition $y = 0$ when considering potential fields in 3D space.

The complex expressions: (1.5) for equipotentials and (1.6) for streamlines can be generalized, respectively, as follows:

$$\varphi_1(x,z,u) = K_1, \quad (2.20)$$

$$\varphi_3(x,z,u)j + \varphi_4(x,z,u)k = K_3 j + K_4 k, \quad (2.21)$$

where $K_1$, $K_3$ and $K_4$ are real constants. The surfaces (2.20) we will call *3D equipotential surfaces*. Expression (2.21) defines the function, which can be regarded as *the function, generating 3D stream surfaces.* The streamlines are situated on imagined 3D stream surfaces.

In the complex analysis a stream function is represented by the imaginary part of complex potential (1.1). The functions $\varphi_3(x,z,u)$ when $u = 0$ and $\varphi_4(x,z,u)$ when $z = 0$ are imaginary parts of relations in complex planes $\xi = x + zj$ and $\xi = x + uk$, respectively, to which we can come if we consider all the possible transitions from the quaternionic space to the complex plane [6]. These functions are the traces of a required stream surface on the $xz$- and $xu$-planes, respectively. Thus, the function, which "generates" a 3D stream surface, means here the function, which contains the traces $\varphi_3(x,z,u)$ when $u = 0$ and $\varphi_4(x,z,u)$ when $z = 0$ on the $xz$- and $xu$-planes.

**Example 2.3.** Consider the quaternionic potential $p^3$ in the Cayley–Dickson doubling form:

$$\psi_H(p) = p^3 = (a + bj)^3 = \phi_1 + \phi_2 j,$$

where

$$\phi_1 = a^3 - (2a + \bar{a})b\bar{b} = \psi_1 + \psi_2 i,$$

$$\phi_2 = (a^2 + a\bar{a} + \bar{a}^2)b - b^2\bar{b} = \psi_3 + \psi_4 i,$$

whence, using (1.12), we obtain

$$\psi_1 = x^3 - 3xy^2 - 3xz^2 - 3xu^2,$$
$$\psi_2 = -y^3 + 3yx^2 - yz^2 - yu^2,$$
$$\psi_3 = -z^3 + 3zx^2 - zy^2 - zu^2,$$
$$\psi_4 = -u^3 + 3ux^2 - uy^2 - uz^2.$$

This function is $\mathbb{H}$-holomorphic because it is constructed from the $\mathbb{C}$-holomorphic function $\xi^3$ by replacing $\xi$ (as a single whole) by $p$ (see Theorem 4.4 in [6]). Hence, it is a quaternionic potential. According to the similarity between rules for differentiation of the $\mathbb{H}$-holomorphic and $\mathbb{C}$-holomorphic functions [6,7,8], we have

$$(p^3)' = 3p^2 = 3(a + bj)^2 = 3(x^2 - y^2 - z^2 - u^2)$$
$$+6xyi + 6xzj + 6xuk.$$

According to (2.11) we also obtain

$$\nabla\psi_1 = \partial_x\psi_1 + \partial_y\psi_1 i + \partial_z\psi_1 j + \partial_u\psi_1 k$$
$$= 3(x^2 - y^2 - z^2 - u^2) - 6xyi - 6xzj - 6xuk.$$

Given (2.19), after the transition $y = 0$, we get

$$(F(p)| = (\overline{(p^3)'}| = 3(x^2 - z^2 - u^2)$$
$$-6xzj - 6xuk = (\nabla\psi_1| = \nabla_3\varphi_1,$$

We see that the quaternionic potential $p^3$ represents a potential field in 3D space with basis vectors $1, j, k$.

We could compute also the gradient $\nabla_3\varphi_1 = \nabla_3(x^3 - 3xz^2 - 3xu^2)$ directly in 3D space.

## 2.2. ℍ-harmonic Constituents of Quaternionic Potentials

In the similar way as in complex analysis [1,4], we define the quaternionic generalizations of the Laplace equations from quaternionic Cauchy-Riemann's equations (1.16). Substituting expressions (1.14), (1.15), (2.9) and (2.10) into (1.16) and then simplifying, we obtain the quaternionic generalization of Cauchy-Riemann's equations in $\mathbb{R}^4$–representation as follows:

$$\begin{aligned}
&1)\ (\partial_x\psi_1 + \partial_y\psi_2 - \partial_z\psi_3 - \partial_u\psi_4 | = 0\\
&2)\ (\partial_z\psi_4 - \partial_u\psi_3 | = 0,\\
&3)\ (\partial_x\psi_2 - \partial_y\psi_1 | = 0,\\
&4)\ (\partial_z\psi_1 + \partial_x\psi_3 | = 0,\\
&5)\ (\partial_u\psi_2 + \partial_y\psi_4 | = 0,\\
&6)\ (\partial_u\psi_1 + \partial_x\psi_4 | = 0,\\
&7)\ (\partial_z\psi_2 + \partial_y\psi_3 | = 0.
\end{aligned} \tag{2.22}$$

Quite analogously we obtain from (2.6) the equations

$$\begin{aligned}
&1)\ \partial_z\psi_4 - \partial_u\psi_3 = 0,\\
&2)\ \partial_y\psi_1 + \partial_x\psi_2 = 0,\\
&3)\ \partial_z\psi_1 + \partial_x\psi_3 = 0,\\
&4)\ \partial_u\psi_1 + \partial_x\psi_4 = 0,\\
&5)\ \partial_u\psi_2 - \partial_y\psi_4 = 0,\\
&6)\ \partial_z\psi_2 - \partial_y\psi_3 = 0,
\end{aligned} \tag{2.23}$$

which unlike equations (2.22) hold for all ℍ-holomorphic functions as early as before the transition $y = 0$. Naturally, they hold true after this transition as well.

To illustrate this we consider, for example, equations (2.22-3) and (2.23-2) with the functions $\psi_1$ and $\psi_2$ from example 2.3. The calculated derivatives are $\partial_x\psi_2 = 6xy$ and $\partial_y\psi_1 = -6xy$. Substituting them into (2.22-3) and (2.23-2) we see that these equations are satisfied:

$$(\partial_x\psi_2 - \partial_y\psi_1| = (6xy + 6xy| = 0$$

after the transition $y = 0$, and

$$\partial_y\psi_1 + \partial_x\psi_2 = -6xy + 6xy = 0$$

without any conditions.

Equations (2.23) complement ℍ-holomorphy equations (2.22). They coincides partly with equations of the modified M. Riesz system ($\mathbb{H}_4$) [9]. However, as shown in [6,7], the systems similar to ($\mathbb{H}_4$) (for the only "left monogenic functions" or only the right those) cannot be essentially complete (essentially adequate). We use the systems of equations (2.22) and (2.23), where each of them unites the left and the right approaches together [6,7], to obtain the quaternionic generalizations of the Laplace equations and harmonic functions.

**Theorem 2.4** Suppose a continuous quaternionic function $\psi_H(p) = \psi_1 + \psi_2 i + \psi_3 j + \psi_4 k$ is ℍ -holomorphic, satisfying equations (2.22) and (2.23) in some open connected set $G_4 \subset \mathbb{H}$. Suppose also that the constituents of this function:

$$\psi_1 = \psi_1(x,y,z,u),\quad \psi_2 = \psi_2(x,y,z,u),$$
$$\psi_4 = \psi_4(x,y,z,u),\quad \psi_3 = \psi_3(x,y,z,u)$$

*possess the continuous second-order non-vanishing derivatives with respect to $x, y, z,$ and $u$ in $G_4$. Then the continuous functions $\psi_1, \psi_2, \psi_3,$ and $\psi_4$ satisfy in $G_4$ the following quaternionic generalization of the Laplace equations*:

$$(\partial_x^2\psi_1 - \partial_y^2\psi_1 + \partial_z^2\psi_1 + \partial_u^2\psi_1| = 0, \tag{2.24}$$
$$(\partial_x^2\psi_2 - \partial_y^2\psi_2 + \partial_z^2\psi_2 + \partial_u^2\psi_2| = 0, \tag{2.25}$$
$$(\partial_x^2\psi_3 - \partial_y^2\psi_3 + \partial_z^2\psi_3 + \partial_u^2\psi_3| = 0, \tag{2.26}$$
$$(\partial_x^2\psi_4 - \partial_y^2\psi_4 + \partial_z^2\psi_4 + \partial_u^2\psi_4| = 0, \tag{2.27}$$

*where the second-order derivatives are denoted by $\partial_\alpha^2$, $\alpha = x, y, z, u$.*

**Proof.** It is evident that if we will differentiate with respect to one of the variables expression (2.22-1), vanishing upon transition $y = 0$, then the derivative of this expression , by virtue of continuity, will also vanish upon this transition when differentiating does not violate the condition of such a transition. In our case of summing the terms in (2.22-1), this condition is equivalent to an existence after differentiating (2.22-1) at least two terms, offsetting each other. The condition of non-vanishing second derivatives suffices for such a goal.

Supposing this, we can differentiate equation (2.22-1) with respect to each of the variables without violating this equation.

Differentiating equation (2.22-1) with respect to $x$, we obtain the following expression:

$$(\partial_x^2\psi_1 + \partial_{xy}\psi_2 - \partial_{xz}\psi_3 - \partial_{xu}\psi_4| = 0. \tag{2.28}$$

Here and beyond, for simplicity, we use the designation $\partial_{\alpha\beta}$ for the second mixed partial derivative with respect to $\alpha$ and $\beta$, $\alpha, \beta = x, y, z, u$; $\partial_\alpha^2 = \partial_{\alpha\alpha}$.

Differentiating equations (2.23-2), (2.23-3), (2.23-4), respectively, with respect to $y, z, u$, we get the following relations:

$$\partial_{xy}\psi_2 = -\partial_y^2\psi_1,\ \partial_{xz}\psi_3 = -\partial_z^2\psi_1,\ \partial_{xu}\psi_4 = -\partial_u^2\psi_1.$$

Substituting these relations into (2.28) we obtain the following equation:

$$(\partial_x^2\psi_1 - \partial_y^2\psi_1 + \partial_z^2\psi_1 + \partial_u^2\psi_1| = 0,$$

which coincides with (2.24). Thus, the validity of equation (2.24) for ℍ-holomorphic functions is proved.

Differentiating (2.22-1) with respect to $y$ results in

$$(\partial_{yx}\psi_1 + \partial_y^2\psi_2 - \partial_{yz}\psi_3 - \partial_{yu}\psi_4| = 0. \tag{2.29}$$

Further, differentiating equation (2.23-2) with respect to $x$, equation (2.23-6) with respect to $z$, and equation (2.23-5) with respect to $u$, we get, respectively:

$$\partial_{yx}\psi_1 = -\partial_x^2\psi_2,\ \partial_{yz}\psi_3 = \partial_z^2\psi_2,\ \partial_{yu}\psi_4 = \partial_u^2\psi_2.$$

Substituting these relations into (2.29) gives the equation

$$(\partial_x^2\psi_2 - \partial_y^2\psi_2 + \partial_z^2\psi_2 + \partial_u^2\psi_2| = 0,$$

which coincides with (2.25). Thus, the validity of equation (2.25) for $\mathbb{H}$-holomorphic functions is proved.

Differentiating (2.22-1) with respect to $z$ results in

$$(\ \partial_{zx}\psi_1 + \partial_{zy}\psi_2 - \partial_z^2\psi_3 - \partial_{zu}\psi_4| = 0. \quad (2.30)$$

Further, differentiating equation (2.23-3) with respect to $x$, equation (2.23-6) with respect to $y$, and equation (2.23-1) with respect to $u$, we obtain respectively:

$$\partial_{zx}\psi_1 = -\partial_x^2\psi_3,\ \partial_{zy}\psi_2 = \partial_y^2\psi_3,\ \partial_{zu}\psi_4 = \partial_u^2\psi_3.$$

Substituting these expressions into (2.30), we get the following equation:

$$(\partial_x^2\psi_3 - \partial_y^2\psi_3 + \partial_z^2\psi_3 + \partial_u^2\psi_3| = 0,$$

which coincides with (2.26). Thus the validity of equation (2.26) for $\mathbb{H}$-holomorphic functions is proved.

The last equation to be proved is equation (2.27). Differentiating (2.22-1) with respect to $u$ we obtain

$$(\ \partial_{ux}\psi_1 + \partial_{uy}\psi_2 - \partial_{uz}\psi_3 - \partial_u^2\psi_4| = 0. \quad (2.31)$$

Further, differentiating equation (2.23-4) with respect to $x$, equation (2.23-5) with respect to $y$ and equation (2.23-1) with respect to $z$, we obtain, respectively,

$$\partial_{ux}\psi_1 = -\partial_x^2\psi_4,\ \partial_{uy}\psi_2 = \partial_y^2\psi_4,\ \partial_{zu}\psi_3 = \partial_z^2\psi_4.$$

Substituting these expressions into (2.31), we obtain the equation

$$(\partial_x^2\psi_4 - \partial_y^2\psi_4 + \partial_z^2\psi_4 + \partial_u^2\psi_4| = 0,$$

coinciding with equation (2.27). The validity of equation (2.27) for $\mathbb{H}$-holomorphic functions is proved. *This completes the proof of theorem 2.4 in whole.*

By analogy with complex analysis [1,11], the functions $\psi_1, \psi_2, \psi_3$ and $\psi_4$, which satisfy quaternionic Laplace's equations (2.24) - (2.27), are called here the $\mathbb{H}$-*harmonic functions*. They are easily constructed from $\mathbb{H}$-holomorphic functions.

**Example 2.5** We show that the components $\psi_1, \psi_2, \psi_3, \psi_4$ of the $\mathbb{H}$-holomorphic function

$$\psi_H(p) = p^3 = (a + bj)^3$$

are $\mathbb{H}$-harmonic. In example 2.3 we have obtained for this function the following expressions:

$$\psi_1 = x^3 - 3xy^2 - 3xz^2 - 3xu^2,$$
$$\psi_2 = -y^3 + 3yx^2 - yz^2 - yu^2,$$
$$\psi_3 = -z^3 + 3zx^2 - zy^2 - zu^2,$$
$$\psi_4 = -u^3 + 3ux^2 - uy^2 - uz^2.$$

The derivatives for equation (2.24) are the following:

$$\partial_x^2\psi_1 = 6x,\ \partial_y^2\psi_1 = -6x,\ \partial_z^2\psi_1 = -6x,\ \partial_u^2\psi_1 = -6x.$$

Substituting these into (2.24), we have

$$(6x + 6x - 6x - 6x| = 0.$$

Hence equation (2.24) is satisfied for the function $p^3$.

The calculated derivatives to be inserted into (2.25) are the following: $\partial_x^2\psi_2 = 6y$, $\partial_y^2\psi_2 = -6y$, $\partial_z^2\psi_2 = -2y$, $\partial_u^2\psi_2 = -2y$. Substituting these into (2.25) we obtain

$$(6y + 6y - 2y - 2y| = (8y| = 0,$$

which is satisfied after the transition $y = 0$.

Analogously one can show that equations (2.26) and (2.27) hold as well.

## 2.3. Uniting of Differential Vector Operations in $\mathbb{H}$ −antiholomorphy Conditions

Just as antiholomorphic functions are defined in complex analysis [5], we can define quaternionic antiholomorphic (briefly, $\mathbb{H}$ −antiholomorphic) functions as functions that are the quaternionic conjugation of the $\mathbb{H}$-holomorphic functions. Replacing (similarly to the complex case) a $\mathbb{H}$-holomorphic function by its quaternionic conjugation, that is, replacing $\phi_1$ by $\overline{\phi_1}$ and $\phi_2$ by $-\phi_2$ in the quaternionic generalization of Cauchy-Riemann's equations (1.16), we can obtain the following equations of $\mathbb{H}$ −antiholomorphy:

$$\begin{cases} 1)\ (\partial_a\overline{\phi_1}| = -(\partial_{\overline{b}}\overline{\phi}_2|, \quad 2)\ (\partial_a\phi_2| = (\partial_{\overline{b}}\phi_1|, \\ 3)\ (\partial_a\overline{\phi_1}| = -(\partial_b\phi_2|, \quad 4)\ (\partial_{\overline{a}}\phi_2| = (\partial_{\overline{b}}\overline{\phi_1}|. \end{cases} \quad (2.32)$$

To rewrite equations (2.32) in $\mathbb{R}^4$ –representation we can use equations (2.22). Denoting a $\mathbb{H}$ −antiholomorphic function by $F(p) = f_1 + f_2 i + f_3 j + f_4 k$, where $f_1 = \psi_1$, $f_2 = -\psi_2$, $f_3 = -\psi_3$, $f_4 = -\psi_4$, and substituting these relations into (2.22), we get the following equations of $\mathbb{H}$ −antiholomorphy in $\mathbb{R}^4$ –representation:

$$\begin{aligned} &1)\ (\partial_x f_1 - \partial_y f_2 + \partial_z f_3 + \partial_u f_4 | = 0 \\ &2)\ (\partial_z f_4 - \partial_u f_3 | = 0, \\ &3)\ (\partial_x f_2 + \partial_y f_1 | = 0, \\ &4)\ (\partial_x f_3 - \partial_z f_1 | = 0, \\ &5)\ (\partial_u f_2 + \partial_y f_4 | = 0, \\ &6)\ (\partial_u f_1 - \partial_x f_4 | = 0, \\ &7)\ (\partial_z f_2 + \partial_y f_3 | = 0. \end{aligned} \quad (2.33)$$

Since the derivative $\psi'_H(p) = \psi'_1 + \psi'_2 i + \psi'_3 j + \psi'_4 k$ is $\mathbb{H}$-holomorphic [6,7,8], its conjugate function – the flow velocity vector $F(p) = \overline{\psi'_H(p)}$ - is a $\mathbb{H}$-antiholomorphic function. If we consider the functions $\psi'_1$, $\psi'_2$, $\psi'_3$, $\psi'_4$ instead of the above functions $\psi_1, \psi_2, \psi_3, \psi_4$, then the functions $f_1 = \psi'_1$, $f_2 = -\psi'_2$, $f_3 = -\psi'_3$, $f_4 = -\psi'_4$ represent the real constituents of a flow velocity vector $F(p) = \overline{\psi'_H(p)}$.

In this case equation (2.33-1) should be interpreted as the desired quaternionic generalization of complex equation (1.10a), that is, as the quaternionic generalization of the equation $div\, F(\xi) = 0$. Equation (2.33-1) holds true (just as equations (2.24) - (2.27)) upon the transition from the quaternionic space to 3D one. However this does not mean that we can directly put $(f_2| = 0$ in these equations (as well as in equations (2.33-3,5,7)) before differentiating.

The function $(\partial_y f_2|$ can be interpreted as some dencity of sources and sinks in 3D space. If $(\partial_y f_2| = 0$, then the potential flow in 3D space is solenoidal.

If we as usual denote the field vector $\vec{F}(x,y,z)$ in 3D space (with the basis vectors $\vec{i},\vec{j},\vec{k}$) by

$$\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k},$$

then the $curl$ of this field is the vector defined [11] by

$$curl\vec{F} = \left(\partial_y R - \partial_z Q\right)\vec{i} + \left(\partial_z P - \partial_x R\right)\vec{j} + \left(\partial_x Q - \partial_y P\right)\vec{k}.$$

Using the above notation $f_1, f_2, f_3, f_4$ for components of the vector $F(p)$ and adapting the last expression to our designations in 3D space (with basis vectors $1, j, k$), viz.:

$$\vec{i} = 1,\ \vec{j} = j,\ \vec{k} = k,\ x = x,\ y = z,\ z = u,$$

$$P = (f_1|,\ Q = (f_3|,\ R = (f_4|,\ \vec{F} = (F(p)|,$$

we obtain the following expression for the $curl\ (F(p)|$ at the points $p \in (G_4|$:

$$curl\left(F\left(p\right)\right| = \left[\partial_z\left(f_4\right| - \partial_u\left(f_3\right|\right] + \left[\partial_u\left(f_1\right| - \partial_x\left(f_4\right|\right]j + \left[\partial_x\left(f_3\right| - \partial_z\left(f_1\right|\right]k \tag{2.34}$$

Comparing this expression with equations (2.33-2), (2.33-6), and (2.33-4), it is now easy to see that the equation $curl\ (F(p)| = 0$ is represented by equations (2.33-2), (2.33-6), and (2.33-4) for the $curl$ components. The quaternionic generalization of the flow velocity divergenz, as noted above, is represented by equation (2.33-1).

*In sum, we have shown that the above-noted property of complex analysis to unite ("condense") vector differential operations (equations) into one system of equations is retained in the presented quaternionic generalization.*

Since we does not take the derivatives of the functions $f_1, f_3, f_4$ with respect to $y$ in equations (2.33-2), (2.33-4), and (2.33-6), the reshuffling of the symbol "(" and any of the symbols "$\partial_x$","$\partial_z$","$\partial_u$" in these equations is correct.

Analogously to the computation of the gradient $(F(p)| = \nabla_3\varphi_1$, the computation of the $curl\ (F(p)|$ can be directly performed in 3D space with basis vectors $1, j, k$, that is, we can first compute $(f_1|$, $(f_3|$, $(f_4|$ and then the $curl\ (F(p)|$ in accordance with (2.34).

When considering potential fluid flows (potential fields) equations (2.33-2), (2.33-4), and (2.33-6) suffice and we shall not dwell further on equation (2.33-1).

It is worth emphasizing that the presented theory describes all the main properties of potential fluid flows in 3D space. Indeed, since it is shown that the property $curl\ (F(p)| = 0$ follows from the considered quaternionic representation of potential fluid flows after the transition to 3D space, one can state that the other known *equivalent properties* [11] of potential flows (fields) follow from this representation as well.

These are the following [11]: a circulation of the velocity vector$(F(p)|$ around any closed contour $L \in (G_4|$ is zero, i.e. $\oint_L\ (F(p)| \cdot d(p| = 0$, where the dot denotes the scalar product, (or, equivalently, the integral of $(F(p)|$ along a path depends only on the endpoints of that path) as well as a possibility to represent the scalar product $(F(p)| \cdot d(p|$ as a total differential of the potential function $(\psi_1| = \varphi_1$.

To illustrate these properties we continue example 2.3.

In considering the quaternionic potential $p^3$, we obtain in 3D space the following expression:

$$(F(p)| = (\overline{(p^3)'}| = (3\overline{p^2}|$$
$$= 3(x^2 - z^2 - u^2) - 6xzj - 6xuk$$
$$= (f_1| + (f_3|j + (f_4|k,$$

whence

$$(f_1| = 3(x^2 - z^2 - u^2),\ (f_3| = -6xz,\ (f_4| = -6xu.$$

Using the expression $(\psi_1| = \varphi_1 = x^3 - 3xz^2 - 3xu^2$ and definition (2.14), we obtain

$$\nabla_3\varphi_1 = \partial_x\varphi_1 + \partial_z\varphi_1 j + \partial_u\varphi_1 k$$
$$= 3(x^2 - z^2 - u^2) - 6xzj - 6xuk.$$

Comparing the obtained expressions for $(F(p)|$ and $\nabla_3\varphi_1$, we see that they coincide, that is, the 3D flow (field) represented by the $\mathbb{H}$-holomorphic function $p^3$ is potential.

By calculating derivatives $\partial_z(f_4|$, $\partial_u(f_3|$, $\partial_u(f_1|$, $\partial_x(f_4|$, $\partial_x(f_3|$, $\partial_z(f_1|$ and substituting them into (2.34), we check that $curl\ (F(p)| = (-6u + 6u)j + (-6z + 6z)k = 0$.

Since the relation $curl\ (F(p)| = 0$ holds in $(G_4|$, the zero circulation of the vector $(F(p)|$ around the boundary $L$ of the surface $S \in (G_4|$ follows from Stokes' theorem [11]:

$$\oint_L\ (F(p)| \cdot d(p| = \iint_S\ curl\ (F(p)| \cdot \vec{n}dS = 0,$$

where

$$(F(p)| = (f_1| + (f_3|j + (f_4|k,\ \ d(p| = dx + dzj + duk,$$

$\vec{n}$ is a unit normal vector to the surface $S$ at any its point, and $dS$ is area of a differential element of $S$.

The equality of $F(p)| \cdot d(p|$ and total differential $d\varphi_1$:

$$(F(p)| \cdot d(p| = (f_1|dx + (f_3|dz + (f_4|du = d\varphi_1$$

is verified by the following relations:

$$(f_1| = 3(x^2 - z^2 - u^2) = \partial_x\varphi_1,$$
$$(f_3| = -6xz = \partial_z\varphi_1,$$
$$(f_4| = -6xu = \partial_u\varphi_1.$$

# 3. The Example of 3D Flow Modeling

To illustrate an application of the presented theory of quaternionic potential to solving problems in 3D space we consider the relatively simple case of quaternionic potential $p^2$ in detail.

By representing this function in the Cayley–Dickson doubling form, we have:

$$\psi_H(p) = p^2 = (a + bj)^2 = \phi_1 + \phi_2 j,$$

where

$$\phi_1 = \left(a^2 - b\overline{b}\right) = \psi_1 + \psi_2 i,$$
$$\phi_2 = b(a + \overline{a}) = \psi_3 + \psi_4 i,$$

whence, using (1.12), we obtain

$$\psi_1 = x^2 - y^2 - z^2 - u^2,$$
$$\psi_2 = 2xy,\quad \psi_3 = 2xz,\quad \psi_4 = 2xu.$$

By the direct calculation we get the following differential expressions:

$$(p^2)' = 2p = 2(a + bj) = 2(x + yi + zj + uk),$$

$$\overline{(p^2)'} = 2\bar{p} = 2(\bar{a} - bj) = 2(x - yi - zj - uk),$$

$$\nabla\psi_1 = \partial_x\psi_1 + \partial_y\psi_1 i + \partial_z\psi_1 j + \partial_u\psi_1 k$$
$$= 2x - 2yi - 2zj - 2uk = \overline{(p^2)'}$$

According to (2.19), after the transition $y = 0$, we get

$$(F(p)| = (\overline{(p^2)'}| = 2(x - zj - uk) = \nabla_3\varphi_1. \quad (3.1)$$

We see that the function $p^2$ represents a potential field in 3D space with orthogonal basis vectors $1, j, k$. In the sequel we retain these designations for 3D basis to emphasize that we deal with 3D relations deduced from the quaternionic expressions. According to (2.20), we get the following equation of 3D equipotential surfaces for the quaternionic potential $p^2$:

$$\varphi_1(x, z, u) = (\psi_1| = x^2 - z^2 - u^2 = K_1 \quad (3.2)$$

The 3D equipotential surfaces represent the two-sheet hyperboloids [10] with the canonical form of the equation

$$\varphi_1 : \frac{z^2}{\alpha^2} + \frac{u^2}{\alpha^2} - \frac{x^2}{\alpha^2} = -1, \quad (3.3)$$

where by $\alpha = \sqrt{K_1}$ are denoted the equal semiaxes. These equipotential surfaces are the surfaces of revolution [10] around the $x$-axis.

The $x$-axis intersects $\varphi_1$ at the points $X_1(\alpha, 0,0)$ and $X_2(-\alpha, 0,0)$, which are the vertexes of hyperboloids. The coordinate axes are axes of symmetry of the hyperboloids, the coordinate planes are the planes of symmetry and origin of coordinates is the center of symmetry.

Now we need to find the stream surfaces $S$ that have traces

$$\varphi_3(x, z) = 2xz = K_3 \quad (3.4)$$

$$\varphi_4(x, u) = 2xu = K_4 \quad (3.5)$$

on the planes $x0z$ and $x0u$, respectively.

Putting $K_3 = K_4 = K$ and rotating counterclockwise the plane $x0z$ around the $x$-axis by the angle $\frac{\pi}{2}$, we can obtain the trace $\varphi_4(x, u)$ from $\varphi_3(x, z)$. Hence the desired stream surfaces $S$ for $p^2$ in the simplest case are the surfaces of revolution around the $x$-axis .

To obtain an equation of such stream surfaces from the equations $\varphi_3(x, z) = 2xz = K$, $y = 0$ we use the standard method [10] of replacing the variable $z$ by $\sqrt{z^2 + u^2}$ in the expression for $\varphi_3(x, z)$:

$$S\colon\ 2x\sqrt{z^2 + u^2} = K$$

or

$$S\colon\ 4x^2(z^2 + u^2) = K^2. \quad (3.6)$$

This equation represents two-sheet hyperboloids in 3D space, corresponding the hyperbolas of streamlines when considering the complex potential $\xi^2$ [1] in the complex planes $\xi = x + zj$ or $\xi = x + uk$.

The stream hyperboloids have the same axes, planes and the center of symmetry as the equipotential hyperboloids. Given this symmetry, we, for simplicity, can consider all graphic representations only in the half-space $x > 0$.

Now we get the equations of tangent planes to surfaces $\varphi_1$ and $S$ at points, belonging to the curve $I$ of intersection of these surfaces. If a surface is defined implicitly by the equation $E(x, z, u) = 0$, then the equation of its tangent plane at the point $(x_0, z_0, u_0) \in E$ is the following [11]:

$$\begin{aligned} \mathrm{T}_E :& (\partial_x E)_0 (x - x_0) + (\partial_z E)_0 (z - z_0) \\ &+ (\partial_u E)_0 (u - u_0) = 0, \end{aligned} \quad (3.7)$$

where by the index 0 are denoted the partial derivatives calculated at the point $(x_0, z_0, u_0)$. The vector of the normal to this surface [11] at that point is

$$\boldsymbol{N_E} = \{(\partial_x E)_0, (\partial_z E)_0, (\partial_u E)_0\}. \quad (3.8)$$

(Here and in the sequel, the designation of a vector by the troika of its components in braces is highlighted in bold).

Using (3.7) we obtain after some calculation the following expressions for planes tangent to surfaces $\varphi_1$ and $S$:

$$\begin{aligned} \mathrm{T}_{\varphi_1} :& -x_0 (x - x_0) + z_0 (z - z_0) \\ &+ u_0 (u - u_0) = 0, (x_0, z_0, u_0) \in \varphi_1, \end{aligned} \quad (3.9)$$

$$\begin{aligned} T_S :& \left(z_0{}^2 + u_0{}^2\right)(x - x_0) + x_0 z_0 (z - z_0) \\ &+ x_0 u_0 (u - u_0) = 0, (x_0, z_0, u_0) \in S. \end{aligned} \quad (3.10)$$

Using (3.8) we get the normal vectors $\boldsymbol{N_{\varphi_1}}$ and $\boldsymbol{N_S}$ to the tangent planes $T_{\varphi_1}$ and $T_S$, respectively, as follows:

$$\boldsymbol{N_{\varphi_1}} = \{-x_0, z_0, u_0,\}, (x_0, z_0, u_0) \in \varphi_1 \quad (3.11)$$

$$\boldsymbol{N_S} = \{z_0{}^2 + u_0{}^2, x_0 z_0, x_0 u_0\}, (x_0, z_0, u_0) \in S. \quad (3.12)$$

Denote by $M_I(x_I, z_I, u_I)$ the points, belonging to the curve $I$ of intersection of the surfaces $\varphi_1$ and $S$. Then, putting $x_0 = x_I$, $z_0 = z_I$, $u_0 = u_I$ in (3.11) and (3.12), we see that the normals $\boldsymbol{N_{\varphi_1}}$ and $\boldsymbol{N_S}$ are perpendicular, since their scalar product at the points $M_I(x_I, z_I, u_I)$ is equal to zero [10,11]:

$$\boldsymbol{N_{\varphi_1}} \cdot \boldsymbol{N_S} = -x_I(z_I{}^2 + u_I{}^2) + x_I z_I{}^2 + x_I u_I{}^2 = 0.$$

*Hence the equipotential surfaces $\varphi_1$ and the stream surfaces $S$ are orthogonal at the points $M_I(x_I, z_I, u_I) \in I$ of their intersection.*

The beginning point $M_I(x_I, z_I, u_I)$ of the velocity vector $(F(p_I)| = 2x_I - 2z_I j - 2u_I k$ lies on the stream surface $S$. To show that $(F(p_I)|$ lies in the plane tangent to $S$ at that point we need to verify that $(F(p_I)|$ is perpendicular to the normal vector $\boldsymbol{N_S}$ at that point. Calculating their scalar product, one can see that it is equal to zero:

$$\begin{aligned} &(F(p_I)| \cdot N_S \\ &= \{2x_I, -2z_I, -2u_I\} \cdot \left\{\left(z_I{}^2 + u_I{}^2\right), x_I z_I, x_I u_I\right\} \\ &= 2x_I\left(z_I{}^2 + u_I{}^2\right) - 2x_I z_I{}^2 - 2x_I u_I{}^2 = 0. \end{aligned}$$

It follows that the vectors $(\boldsymbol{F(p_I)}|$ and $\boldsymbol{N_S}$ at the point $M_I$ are perpendicular. Hence the flow velocity vector

$(F(p_I)|$ lies in the plane tangent to stream surface $S$ at the point $M_I$.

Now we consider the plane $\lambda$ passing through the point $M_2(x_2, z_2, u_2), x_2, z_2, u_2 \neq 0$, and the $x$-axis. This plane is the same as the plane passing through the three points $M_0 = O(0,0,0)$, $M_1(x_1, 0,0), x_1 \neq 0$, and $M_2(x_2, z_2, u_2)$. By adapting the general equation of a plane passing through three given points in space [11] to designations of the points $M_0$, $M_1$ and $M_2$ in our system of coordinates $x, z, u$, we get the following equation:

$$\begin{vmatrix} x - x_0 & z - z_0 & u - u_0 \\ x_1 - x_0 & z_1 - z_0 & u_1 - u_0 \\ x_2 - x_0 & z_2 - z_0 & u_2 - u_0 \end{vmatrix} = \begin{vmatrix} x & z & u \\ x_1 & 0 & 0 \\ x_2 & z_2 & u_2 \end{vmatrix} = 0,$$

whence further follows the equation of the plane $\lambda$, passing through the point $M_2(x_2, z_2, u_2)$ and the $x$-axis:

$$\lambda: \quad 0 \cdot x - u_2 \cdot z + z_2 \cdot u = 0.$$

The normal vector to the plane $\lambda$ is

$$\boldsymbol{N_\lambda} = \{0, -u_2, z_2,\}. \tag{3.13}$$

Putting $z_2 = z_I$ , $u_2 = u_I$ in (3.13), we obtain the following expression for the normal vector $\boldsymbol{N_\lambda}$ at the point $M_I$:

$$\boldsymbol{N_\lambda} = \{0, -u_I, z_I\}. \tag{3.14}$$

Calculating the scalar product of the vectors $(\boldsymbol{F}(\boldsymbol{p_I})| = \{2x_I, -2z_I, -2u_I\}$ and $\boldsymbol{N_\lambda}$, we get that it is equal to zero:

$$(\boldsymbol{F}(\boldsymbol{p_I})| \cdot \boldsymbol{N_\lambda} = \{2x_I, -2z_I, -2u_I\} \cdot \{0, -u_I, z_I\}$$
$$= 2z_I u_I - 2u_I z_I = 0.$$

Thus, we have now established that the flow velocity vector $(F(p_I)|$ at the point $M_I(x_I, z_I, u_I) \in I$ lies in the plane $\lambda$.

*Ultimately, we have shown that the flow velocity vector* $(F(p_I)|$ *at the point* $M_I(x_I, z_I, u_I) \in I$ *lies on the line of intersection of the planes* $T_S$ *and* $\lambda$ *passing through this point*. Since each point of the stream surface $S$ can be interpreted as a point $M_I$ of the intersection of this surface with the corresponding equipotential surface $\varphi_1$ (by the appropriate choice of the constant $K_1$ for given $K$), this result holds true for all points of the stream surface S.

Putting $x = x_I$ , $z = z_I$ and $u = u_I$ in equations (3.2) and (3.6), we establish the equation of the intersection curve $I$, which is the circle of radius $r_I$ with the center $(x_I, 0,0)$:

$$z_I^{\ 2} + u_I^{\ 2} = r_I^2 = x_I^2 - K_1 = \frac{K^2}{4x_I^2}. \tag{3.15}$$

Figure 1 demonstrates two orthogonal surfaces $\varphi_1$ and $S$ computed by using formulas (3.2) and (3.6), respectively.

Given the above-mentioned symmetry, the surfaces are depicted only in the half-space $x > 0$. According to (3.15) for $K_1 = 5$ and $K = 12$, the intersection curve of depicted surfaces is the circle of radius $r_I = 2$ with the center $(3,0,0)$. The flow velocity vectors $(F(p)|$ indicated with arrows on the surface $S$ are computed by using formula (3.1). The sizes of the depicted arrows depend on the absolute values $2\sqrt{x^2 + z^2 + u^2}$ of the flow velocities at the points of $S$. The arrows represent the so-called slice (the part) of the flow velocities vector space by the slice surface $S$. Analogously one can plot other couples of the orthogonal surfaces $\varphi_1$ and $S$.

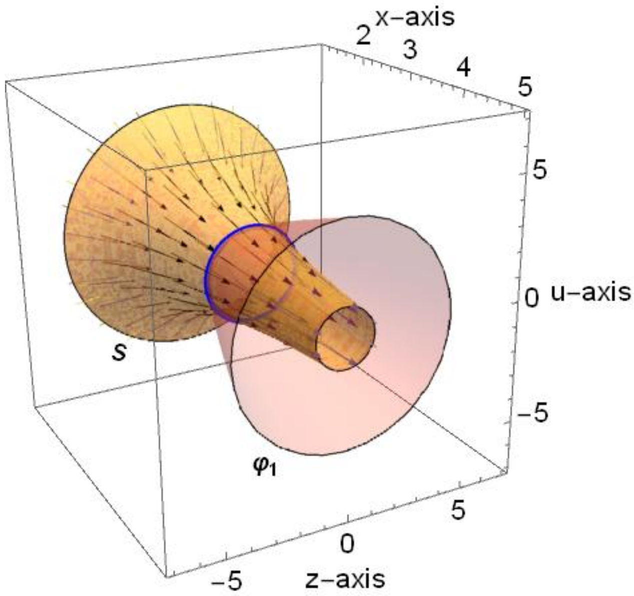


**Figure 1.** The quaternionic potential $p^2$. Example of orthogonal stream and potential surfaces: $S$ for $K = 12$ and $\varphi_1$ for $K_1 = 5$. The intersection curve is highlighted by blue colour.

By creating this drawing the computing system Wolfram Mathematica® [12] was used.

In general, the presented conception of the quaternionic potential allows us to investigate a large manifold of 3D potential functions and stationary fields, corresponding a large number of $\mathbb{H}$-holomorphic functions constructed from $\mathbb{C}$-holomorphic functions.

## References


[1] Mathews, J. H., Howell, R. W., *Complex Analysis for Mathematics and Engineering*, 3rd ed, Jones and Bartlett Publishers, Boston-Toronto-London-Singapore, 1997.

[2] Morse, P. M., Feshbach, H., *Methods of theoretical physics*, Part 1: Chapters 1 to 8, McGraw-Hill Book Co., Inc., New York-Toronto-London, 1953.

[3] Rönn, S., *Bicomplex algebra and function theory*, arXiv: math.CV/ 0101200v1, Jan 2001.

[4] Gürlebeck, K., Habetta, K., Sprößig, W., *Funktionentheorie in der Ebene und im Raum*, Birkhäuser Verlag, Basel-Boston-Berlin, 2006.

[5] Remmert, R., *Theory of Complex Functions*, Springer-Verlag, New York Inc, 1991.

[6] Parfenov, M., *Adequate quaternionic generalization of complex differentiability*, [viXra: Functions and Analysis], Aug 2016. Available: vixra.org/abs/1609.0006

[7] Parfenov, M., "On Properties of Holomorphic Functions in Quaternionic Analysis." *American Journal of Mathematical Analysis*, Vol. 5, No. 1, July 2017, pp. 17-24. Available: pubs.sciepub.com/ajma/5/1/4/index.html

[8] Parfenov, M., *The Similarity between Rules for Essentially Adequate Quaternionic and Complex Differentiation*, [viXra: Functions and Analysis], June 2018. Available: vixra.org/abs/1806.0239

[9] Leutwiler, H., Zeilinger, P., "On Quaternionic Analysis and its Modifications", *Computational Methods and Function Theory,* 4 (1), 159-182, Aug 2004.

[10] Podoksenov, M. N., Prokhozhiy, S. A., *Analytic Geometry in the space*, "VSU named after P.M.Masherov" Publishers, Vitebsk, 2013.

[11] Polyanin, A. D., Manzhirov, A. V., *Handbook of Mathematics for Engineers and Scientists,* Chapman & Hall/CRC, Boca Raton-London-New York, 2007.

[12] Wellin, P.R., Gaylord, R. J., Kamin, S.N., *An Introduction to Programming with Mathematica*, 3rd ed, Cambridge University Press, New York, 2005.