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α-β-ψ-φ Contraction in Digital Metric Spaces

Ibtisam A. Masmali[1], Ghaliah Y. Alhamzi[2], Sumitra Dalal[1,*]

[1]Jazan University, Jazan, K.S.A.
[2]Al Imam Mohammad Ibn Saud Islamic University, Riyadh, K.S.A.
*Corresponding author: mathsqueen_d@yahoo.com

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Abstract Samet et al. (Nonlinear Anal. 75, 2012, 2154-2165) introduced a new, simple and unified approach by using the concepts of α-ψ-contractive type mappings and α-admissible mappings in metric spaces and presented some nice fixed point results. Recently, Sridevi et.al (International Journal of Mathematics Trends and Technology, Volume 48, Number 3 August 2017) proposed the concept of $\alpha-\psi-\varphi$ contraction and generalized $\alpha-\psi-\varphi$ for self map in digital metric spaces. The purpose of this paper is to present a new class of contractive pair of mappings called $\alpha-\beta-\psi-\varphi$ contraction and generalized $\alpha-\beta-\psi-\varphi$ contractive pair of mappings and study various fixed point theorems for such mappings in digital metric spaces. For this, we introduce a new notion of $\alpha-\beta$-admissible w.r.t T mapping which in turn generalizes the concept of g-monotone mapping recently given by "Ciric et al. (Fixed Point Theory Appl. 2008 (2008), Article ID 131294, 11 pages)". Also, we give some fixed point theorems for cyclic contractive mapping in such spaces. The presented theorems hold without using completeness of the space and without the assumption of continuity of the given mappings. Our results extend, generalize and subsumes digital version of various known comparable results [[1-4,8,13,16,18-22], worth to mention here]. Some illustrative examples are quoted to demonstrate the main results.

Keywords: *Digital image, Digital metric space, Banach contractive principle, $\alpha-\beta$ -admissible maps, $\alpha-\beta-\psi-\varphi$ contraction and generalized $\alpha-\beta-\psi-\varphi$ contraction*

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1. Introduction

In metric fixed point theory the contractive conditions on underlying functions play an important role for finding solution of fixed point problems. Banach contraction principle [8] is a fundamental result in metric fixed point theory. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle.

Recently, Rosenfeld [17] proposed an impressive generalizations of the notion of a metric as digital metric. Digital topology is the study of the topological properties of images arrays. The results provide a sound mathematical basis for image processing operations such as image thinning, border following, contour filling and object counting. Digital topology is important for computer vision, image processing and computer graphics. Kong [15], then introduced the digital fundamental group of a discrete object. The digital versions of the topological concepts were given by Boxer [4], who later studied digital continuous functions [5]. Later, he established results of digital homology groups of 2D digital images in [6]. Ege and Karaca [9,10,11,12] give relative and reduced Lefschetz fixed point theorem for digital images.

Samet et al. [19] extended and generalized the Banach contraction principle by introducing a new category of contractive type mappings known as $\alpha-\psi$ contractive type mapping. Further, Karapinar and Samet [16], Gulyaz [13], Bilgili [3], Bota [4] generalized the $\alpha-\psi$ -contractive type mappings and obtained various fixed point theorems for this generalized class of contractive mappings. Recently, Sridevi et. al [22], introduced digital $\alpha-\psi-\varphi$ -contraction and proved fixed point results for self maps in digital metric spaces.

In this paper we generalize the concept of $\alpha-\psi$ -contractive mappings in the setting of digital metric space, as digital- $\alpha-\beta-\psi-\varphi$ -contractive mappings. We introduce a new notion of $\alpha-\beta$ -admissible w.r.t T mapping and establish some coincidence and common fixed point theorems for the generalized digital- $\alpha-\beta-\psi-\varphi$ -contractive pair of mappings. Our results unify and generalize the results derived by Sridevi et.al [22], Karapinar and Samet [16], Gulyaz [13], Bilgili [3], Bota [4] and various other related results in the literature. Also, we furnish some non-trivial examples to elicit the usability of the obtained theorems.

2. Preliminaries

This document unfolds with preliminaries section, where we review some definitions, examples and notable results that are involved in the sequel. Before referring to the works, first of all, we need to recall the following:

Let X be subset of Z^n for a positive integer n where Z^n is the set of lattice points in the n- dimensional Euclidean space and ρ represent an adjacency relation for the members of X. A digital image consists of (X, ρ).

Definition 2.1 [7]: Let l, n be positive integers, $1 \le l \le n$ and two distinct points

$$a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n) \in Z^n,$$

a and b are k_l - adjacent if there are at most l indices i such that $|a_i - b_i| = 1$ and for all other indices j such that $|a_j - b_j| \neq 1, a_j = b_j$.

A ρ -neighbour [7] of $a \in Z^n$ is a point of Z^n that is ρ - adjacent to a where $\rho \in \{2,4,6,8,18,26\}$ and $n \in \{1,2,3\}$. The set

$N_\rho(a) = \{b / \ b \text{ is adjacent to } a\}$

is called the ρ - neighbourhood of a. A digital interval [14] is defined by $[p,q]_z = \{z \in Z / p \le z \le q\}$, where $p, q \in Z$ and $p < q$.

A digital image $X \subset Z^n$ is ρ - connected [3] if and only if for every pair of different points $u, v \in X$, there is a set $\{u_0, u_1, ..., u_r\}$ of points of digital image X such that $u = u_0, v = u_r$ and u_i and u_i and u_{i+1} are ρ - neighbours where $i = 0,1,2,...,r-1$.

Definition 2.2: [7] Let $(X, \rho_0) \subset Z^{n_0}, (Y, \rho_1) \subset Z^{n_1}$ be digital images and $T : X \to Y$ be a function, then

i) T is said to be (ρ_0, ρ_1) - continuous [7], if for all ρ_0 - connected subset E of X, $f(E)$ is a ρ_1 - connected subset of Y.

ii) For all ρ_0 - adjacent points $\{u_0, u_1\}$ of X, either $T(u_0) = T(u_1)$ or $T(u_0)$ and $T(u_1)$ are a ρ_1 - adjacent in Y if and only if T is (ρ_0, ρ_1) - continuous .

iii) If T is (ρ_0, ρ_1) - continuous, bijective and T^{-1} is (ρ_0, ρ_1) - continuous, then T is called (ρ_0, ρ_1) - isomorphism [6] and denoted by $X \cong_{(\rho_0, \rho_1)} Y$.

Definition 2.3 [22] Let $X \subset Z^n$, d be the Euclidean metric on Z^n, (X, d) is a metric space. Suppose (X, ρ) is a digital image with ρ - adjacency then (X, d, ρ) is called a digital metric space.

Definition 2.4 [22]: A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) is a Cauchy sequence if for all $\varepsilon > 0$, there exists $\delta \in N$ such that for all $n,m > \delta$, then $d(x_n, x_m) < \varepsilon$.

Definition 2.5 [22]: A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) converges to a limit $p \in X$ if for all $\varepsilon > 0$, there exists $\alpha \in \mathbb{N}$ such that for all $n > \delta$, then $d(x_n, p) < \varepsilon$.

Definition 2.6 [22]: A digital metric space (X, d, ρ) is a complete digital metric space if any Cauchy sequence $\{x_n\}$ of points of (X, d, ρ) converges to a point p of (X, d, ρ).

Definition 2.7 [22]: Let (X, d, ρ) be any digital metric space and $T : (X, d, \rho) \to (X, d, \rho)$ be a self digital map. If there exists $\alpha \in (0,1)$ such that for all $x \in X$, $d(Tx, Ty) \le \alpha(x, y)$, then T is called a digital contraction map.

Proposition 2.8 [7]: Every digital contraction map is digitally continuous.

Theorem 2.9 [7]: (Banach Contraction principle) Let (X, d, ρ) be a complete digital metric space which has a usual Euclidean metric in Z^n. Let $T : X \to X$ be a digital contraction map. Then T has a unique fixed point, i.e. there exists a unique $p \in X$ such that $T(p) = p$.

Definition 2.10 [22] Let T be a self mapping on X and let $\alpha : X \times X \to [0, \infty)$ be a function. We say T is an α -admissible mapping if

$$x, y \in X, \alpha(x, y) \ge 1 \Rightarrow \alpha(Tx, Ty) \ge 1.$$

Let $\phi = \{\varphi : [0, \infty) \to [0, \infty)\}$ be such that φ is increasing, $\varphi(t) < t$ and $\varphi(t) = 0$ iff $t = 0$.

Definition 2.11 [22] Let (X, d, ρ) be a digital metric space and $T : X \to X$ and $\varphi \in \phi.$ Suppose $d(Tx, Ty) \le \varphi(d(x, y))$ for all $x, y \in X$. Then T is called a digital φ contraction.

Definition 2.12 [22] Let (X, d, ρ) be a digital metric space and $T : X \to X$ be a digital $\alpha - \psi - \varphi$ -contractive type mapping if there exist functions $\alpha : X \times X \to [0, \infty)$ and $\psi, \varphi \in \phi$ such that

$$\alpha(x, y)\psi(d(Tx, Ty)) \\ \le \psi(d(x, y)) - \varphi(d(x, y)), \forall x, y \in X.$$

Sridevi et.al [22] proved the following results:

Theorem 2.1 [22] Let (X, d, ρ) be a digital metric space and $T : X \to X$ be a digital $\alpha - \psi - \varphi$ -contractive type mapping. Suppose T satisfies the following conditions

i) T is α -admissible ;

ii) There exist $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;

Then

i) T has a fixed point.

ii) If further u, v are fixed points of T with $\alpha(u, v) \ge 1$, then $u = v$

Theorem 2.2 [22] Let (X,d,ρ) be a digital metric space and $T:X\to X$ be a digital $\alpha-\psi-\varphi$ -contractive type mapping. Suppose T satisfies the following conditions

i) T is α -admissible ;

ii) There exist $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$;

iii) T is digital continuous.

iv)

$$\alpha(x,y)\psi(d(Tx,Ty))\le\psi(M(x,y))-\varphi(M(x,y)),$$

where

$$M(x,y)=\max\left\{\begin{matrix} d(x,y),d(x,Tx),d(y,Ty), \\ \frac{1}{2}\left[d(x,Tx)+d(y,Ty)\right]\end{matrix}\right\}.$$

For all $x,y\in X$. Then T has a fixed point.If further u,v are fixed points of T with $\alpha(u,v)\ge 1$, then $u=v$. It is in this case T has unique fixed point.

3. Main Results

We start the main section by introducing the new concepts of $\alpha-\beta$-admissible mapping, $\alpha-\beta$-admissible w.r.t T mapping, $\alpha-\beta-\psi-\varphi$ - contractive and generalized $\alpha-\beta-\psi-\varphi$ - contractive pair of mappings.

Definition 3.1 Let T be a self mapping on X and let $\alpha,\beta:X\times X\to[0,\infty)$ be a function. We say T is an $\alpha-\beta$-admissible mapping if $\alpha(x,y)\ge 1$ and $\beta(x,y)\ge 1$ implies $\alpha(Tx,Ty)\ge 1$ and $\beta(Tx,Ty)\ge 1$ for all $x,y\in X$.

Example Let $X=[0,\infty)$. Define the mapping $S:X\to X$ and $\alpha,\beta:X\times X\to[0,\infty)$ by

$$\alpha(x,y)=\left\{\begin{matrix}3, x>y\\ 0, otherwise\end{matrix}\right\},$$

$$\beta(x,y)=\left\{\begin{matrix}5, x>y\\ 0.5, otherwise\end{matrix}\right\}, Sx=x^2,$$

for all $x,y\in X$.

Now we prove that S is $\alpha-\beta$ -admissible. Let $\alpha(x,y)\ge 1$ and $\beta(x,y)\ge 1$ implies that

$$x>y\Rightarrow x^2>y^2\Rightarrow Sx>Sy\Rightarrow\alpha(Sx,Sy)\ge 1$$

and $\beta(Sx,Sy)\ge 1$.

Definition 3.2 Let (X,d,ρ) be a digital metric space and $T:X\to X$ be a digital $\alpha-\beta-\psi-\varphi$ -contractive type mapping if there exist functions $\alpha,\beta:X\times X\to[0,\infty)$ and $\psi,\varphi\in\phi$ such that

$$\alpha(x,Tx)\beta(y,Ty)\psi(d(Tx,Ty))$$
$$\le\psi(d(x,y))-\varphi(d(x,y)),\forall x,y\in X.$$

Definition 3.3 Let (X,d,ρ) be complete digital metric space and $T:X\to X$ and $\alpha:X\times X\to[0,\infty)$ be the mappings. X is α -regular if $\{x_n\}$ is a sequence in X such that $x_n\to x\in X,\alpha(x_n,x_{n+1})\ge 1$ for all n, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x_{n_{k+1}})\ge 1$ for all $k\in N$ and $\alpha(x,Tx)\ge 1$.

Definition 3.4 Let (X,d,ρ) be complete digital metric space and $T:X\to X$ and $\alpha,\beta:X\times X\to[0,\infty)$ be the mappings. X is $(\alpha-\beta)$ regular if $\{x_n\}$ is a sequence in X such that $x_n\to x\in X,\alpha(x_n,x_{n+1})\ge 1,\beta(x_n,x_{n+1})\ge 1$ for all n, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x_{n_{k+1}})\ge 1,\beta(x_{n_k},x_{n_{k+1}})\ge 1$ for all $k\in N$ and $\alpha(x,Tx)\ge 1,\beta(x,Tx)\ge 1$.

Theorem 3.1 Let (X,d,ρ) be complete digital metric space and $T:X\to X$ and $\alpha,\beta:X\times X\to[0,\infty)$ be the mappings satisfying the following conditions:

i) T is $\alpha-\beta$-admissible ;

ii) There exist $x_0\in X$ such that $\alpha(x_0,Tx_0)\ge 1$ and $\beta(x_0,Tx_0)\ge 1$

iii) Either T is continuous or X is $(\alpha-\beta)$ regular

iv)

$$\alpha(x,Tx)\beta(y,Ty)\psi d(Tx,Ty)(\ )$$
$$\le\psi(d(x,y))-\varphi(d(x,y)),\forall x,y\in X,$$

and $\psi,\varphi\in\phi$. Then T has a fixed point. Further if u,v are fixed points of T with $\alpha(u,Tu)\ge 1,\alpha(v,Tv)\ge 1$ and $\beta(u,Tu)\ge 1,\beta(v,Tv)\ge 1$, then $u=v$.

Proof: Let $x_0\in X$ be such that $\alpha(x_0,Tx_0)\ge 1$ and $\beta(x_0,Tx_0)\ge 1$. Now we can construct a sequence x_n in X by

$$x_{n+1}=Tx_n=T^{n+1}x \text{ for all } n\ge 0. \quad (1)$$

Moreover, we assume that if $x_{n_0+1}=x_{n_0}$ for some $n_0\in N$, then n_0 is a fixed point of T. Consequently, we suppose that $x_{n+1}\ne x_n$ for all $n\in N$. Since T is $\alpha-\beta$-admissible, so

$$\alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\ge 1,$$
$$\alpha(Tx_1,Tx_2)=\alpha(x_2,x_3)\ge 1,$$

and by induction we have $\alpha(x_n,x_{n+1})\ge 1$ for all $n\in N$. Similarly, we have $\beta(x_n,x_{n+1})\ge 1$ for all $n\in N$.

From (iii),

$$\begin{aligned}\psi(d(x_{n+1},x_{n+2}))&=\psi(d(Tx_n,Tx_{n+1}))\\ &\le\alpha(x_n,Tx_n)\beta(x_{n+1},Tx_{n+1})\psi(Tx_n,Tx_{n+1})\\ &\le\psi(d(x_n,x_{n+1}))-\varphi(d(x_n,x_{n+1}))\\ &\le\psi(d(x_n,x_{n+1})).\end{aligned} \quad (2)$$

As ψ is non decreasing, we get $\{d(x_n, x_{n+1}) : n \geq 0\}$ is a non-increasing sequence. Hence there is $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Letting $n \to \infty$ in (2), we have $\psi(r) \leq \psi(r) - \varphi(r) \Rightarrow \varphi(r) = 0$ and hence $r = 0$.

Step 2 We will prove that $\{x_n\}$ is a Cauchy sequence. Suppose to contrary; that is, $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon. \tag{3}$$

This means that

$$d(x_{n_k-1}, x_{m_k}) < \varepsilon. \tag{4}$$

From (3) and (4) and triangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k}) \\ &< \varepsilon + d(x_{n_k}, x_{n_k-1}). \end{aligned} \tag{5}$$

On letting $k \to \infty$ in above inequality and using (3), we have

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon. \tag{6}$$

Also,

$$\begin{aligned} \varepsilon &\leq d(x_{n_k}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \end{aligned}$$

and

$$\begin{aligned} & d(x_{n_k-1}, x_{m_k-1}) \\ &\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_k-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} & d(x_{n_k}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \\ & 2d(x_{n_k}, x_{n_k-1}) + d(x_{n_k}, x_{m_k}) + 2d(x_{m_k}, x_{m_k-1}). \end{aligned} \tag{7}$$

Using (4), (6) and (7) and $k \to \infty$, we get

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k-1}) = \varepsilon. \tag{8}$$

On putting $x = x_{n_k}$ and $y = x_{m_k}$ in (iii), we have

$$\begin{aligned} \psi(d(x_{n_k}, x_{m_k})) &= \psi(d(Tx_{n_k-1}, Tx_{m_k-1})) \\ &\leq \alpha(x_{n_k-1}, Tx_{n_k-1}) \beta(x_{m_k-1}, Tx_{m_k-1}) \\ &\quad \times \psi(d(Tx_{n_k-1}, Tx_{m_k-1})) \\ &\leq \psi(d(x_{n_k-1}, x_{m_k-1})) - \varphi(d(x_{n_k-1}, x_{m_k-1})). \end{aligned} \tag{9}$$

Letting $k \to \infty$, in (9) and using continuity of ψ, we obtain $\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon) \Rightarrow \varphi(\varepsilon) = 0 \Rightarrow \varepsilon = 0$, a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists $x \in X$, such that $x_n \to x$.

First, we suppose that T is continuous, then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tx.$$

Now, suppose that X is $(\alpha - \beta)$ regular. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k-1}, x_{n_k}) \geq 1$ and $\beta(x_{n_k-1}, x_{n_k}) \geq 1$ for all $k \in N$ and $\alpha(x, Tx) \geq 1$ and $\beta(x, Tx) \geq 1$. Using (iii) for $x = x_{n_k}, y = x$, we get

$$\begin{aligned} \psi(d(x_{n_k+1}, Tx)) &= \psi(d(Tx_{n_k}, Tx)) \\ &\leq \alpha(x_{n_k}, Tx_{n_k}) \beta(x, Tx) \psi(d(Tx_{n_k}, Tx)) \\ &\leq \psi(d(x_{n_k}, x)) - \varphi(d(x_{n_k}, x)). \end{aligned}$$

Letting $k \to \infty$, we get $Tx = x$. Further, suppose x' and y' be two fixed points of T such that $\alpha(x', Tx') \geq 1, \alpha(y', Ty') \geq 1$ and

$$\beta(x', Tx') \geq 1, \beta(y', Ty') \geq 1.$$

Again from (iii), we have

$$\begin{aligned} \psi(d(x', y')) &= \psi(d(Tx', Ty')) \\ &\leq \alpha(x', Tx') \beta(y', Ty') \psi(d(Tx', Ty')) \\ &\leq \psi(d(x', y')) - \varphi(d(x', y')) \Rightarrow x' = y'. \end{aligned}$$

Hence T has unique fixed point.

Definition 3.5 Let $S, T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. We say that S is $\alpha - \beta$-admissible w.r.t T if for all $x, y \in X$, we have $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$ implies that $\alpha(Sx, Sy) \geq 1$ and $\beta(Sx, Sy) \geq 1$.

Example: Let $X = [0, \infty)$. Define the mappings $S, T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{Bmatrix} 3, x > y \\ 0, otherwise \end{Bmatrix},$$

$$\beta(x, y) = \begin{Bmatrix} 1, x > y \\ 0.2, otherwise \end{Bmatrix}, Sx = x^2, Tx = 2x,$$

for all $x, y \in X$.

Now we prove that S is $\alpha - \beta$-admissible w.r.t T. Let $\alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$ implies that

$$2x > 2y \Rightarrow x > y \Rightarrow x^2 > y^2 \Rightarrow Sx > Sy \Rightarrow \alpha(Sx, Sy) \geq 1$$

and $\beta(Sx, Sy) \geq 1$ and $\beta(Sx, Sy) \geq 1$.

Remark: Clearly, every α -admissible mapping is $\alpha - \beta$ -admissible w.r.t T mapping when $T = I$. The following example shows that a mapping which is $\alpha - \beta$ -admissible w.r.t T may not be α -admissible.

Example: Let $X = [0, \infty)$. Define the mappings $S, T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{Bmatrix} 3, x > y \\ 0, otherwise \end{Bmatrix},$$

$$\beta(x, y) = \begin{Bmatrix} 1, x > y \\ 0.5, otherwise \end{Bmatrix}, Tx = e^{-x}, Sx = \frac{1}{x},$$

for all $x, y \in X$.

Suppose $\alpha(x, y) \geq 1$, $\beta(x, y) \geq 1$, then we get $\frac{1}{x} < \frac{1}{y} \Rightarrow \alpha(Sx, Sy) = 0 < 1$, which shows S is not α -admissible. Now we prove that S is $\alpha - \beta$ -admissible w.r.t T . Let

$$\alpha(Tx, Ty) \geq 1 \Rightarrow Tx > Ty$$
$$\Rightarrow e^{-x} > e^{-y} \Rightarrow \frac{1}{x} > \frac{1}{y} \Rightarrow \alpha(Sx, Sy) \geq 1.$$

Therefore, S is α -admissible w.r.t T .

Definition 3.5 Let (X, d, ρ) be a digital metric space and $S, T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$ be the mappings. Then we say that the pair (S, T) satisfy $\alpha - \beta - \psi - \varphi$ -contractive type mapping if

$$\alpha(x, Tx)\beta(y, Ty)\psi(d(Sx, Sy)) \leq \psi(d(Tx, Ty)) - \varphi(d(Tx, Ty)), \quad (10)$$

$\forall x, y \in X$ and $\psi, \varphi \in \phi$.

Theorem 3.2 Let (X, d, ρ) be a complete digital metric space and $S, T : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$ be $\alpha - \beta - \psi - \varphi$ mappings satisfying the following:

i) $S(X) \subseteq T(X)$;

ii) S is $\alpha - \beta$ -admissible w.r.t T ;

iii) There exist $x_0 \in X$ such that $\alpha(Tx_0, Sx_0) \geq 1$ and $\beta(Tx_0, Sx_0) \geq 1$;

iv) $$\alpha(x, Tx)\beta(y, Ty)\psi(d(Sx, Sy)) \leq \psi(d(Tx, Ty)) - \varphi(d(Tx, Ty))$$

v) If $\{Tx_n\}$ is a sequence in X such that $\alpha(Tx_n, Tx_{n+1}) \geq 1$ and $\beta(Tx_n, Tx_{n+1}) \geq 1$ for all n and $Tx_n \to Tx \in T(X)$ as $n \to \infty$, then there exists a subsequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $\alpha(Tx_{n(k)}, Tz) \geq 1$ and $\beta(Tx_{n(k)}, Tz) \geq 1$ for all k.

Also suppose $T(X)$ is closed. Then S and T have a coincide point.

Proof In view of condition, (iii) let $x_0 \in X$ be such that $\alpha(Tx_0, Sx_0) \geq 1$ and $\beta(Tx_0, Sx_0) \geq 1$. Since $S(X) \subseteq T(X)$, we can choose a point $x_1 \in X$ such that $Sx_0 = Tx_1$. Continuing this process having chosen $x_1, x_2, ...x_n$ we choose x_{n+1} in X such that

$$x_n = Sx_n = Tx_{n+1}, n = 0, 1, 2, ... \quad (11)$$

We complete the proof in following steps.

Step 1 First we prove that

$$\lim_{n \to \infty} d(Sx_n, Sx_{n+1}) = 0. \quad (12)$$

Since S is $\alpha - \beta$ -admissible w.r.t T, we have

$$\alpha(Tx_0, Sx_0) = \alpha(Tx_0, Tx_1) \geq 1$$
$$\Rightarrow \alpha(Sx_0, Sx_1) = \alpha(Tx_1, Tx_2) \geq 1.$$

Using mathematical induction, we get

$$\alpha(Tx_n, Tx_{n+1}) \geq 1, \forall n = 0, 1, 2, ... \quad (13)$$

Similarly, we can prove that,

$$\beta(Tx_n, Tx_{n+1}) \geq 1, \forall n = 0, 1, 2, ...$$

If $Sx_n = Sx_{n+1}$ for some n , then by (11),

$$Tx_{n+1} = Sx_{n+1}, n = 0, 1, 2, ...,$$

that is S and T have a coincide point at $x = x_{n+1}$ and hence the proof. For this, we suppose that $d(Sx_n, Sx_{n+1}) > 0$ for all n. Applying (10)and (13),

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &= \psi(d(Sx_n, Sx_{n+1})) \\ &\leq \alpha(x_n, Tx_n)\beta(x_{n+1}, Tx_{n+1})\psi(d(Sx_n, Sx_{n+1})) \\ &\leq \psi(d(Tx_n, Tx_{n+1})) - \varphi(d(Tx_n, Tx_{n+1})) \\ &= \psi(d(Sx_{n-1}, Sx_n)) - \varphi(d(Sx_{n-1}, Sx_n)) \\ &\leq \psi(d(Sx_{n-1}, Sx_n)) \end{aligned} \quad (14)$$

As ψ is non decreasing, we get

$$\{d(Sx_n, Sx_{n+1}) : n \geq 0\}$$

is a non-increasing sequence. Hence there is $r \geq 0$ such that $\lim_{n \to \infty} d(Sx_n, Sx_{n+1}) = r$. Letting $n \to \infty$ in (14), we have $\psi(r) \leq \psi(r) - \varphi(r) \Rightarrow \varphi(r) = 0$ and hence $r = 0$.

Step 2: We will prove that $\{Sx_n\}$ is a Cauchy sequence. From (12), it is sufficient to show that $\{Sx_{2n}\}$ is a Cauchy sequence. Suppose to contrary; that is, $\{Sx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two sub sequences of positive integers $(m(i))$ and $(n(i))$ such that $n(i)$ is smallest index for which

$$n(i) > m(i) > i, d(Sx_{2m(i)}, Sx_{2n(i)}) \geq \varepsilon. \quad (15)$$

This means that

$$d\left(Sx_{2m(i)}, Sx_{2n(i)-2}\right) < \varepsilon. \quad (16)$$

From (14)and (16) and triangular inequality, we get

$$\varepsilon \le d\left(Sx_{2m(i)}, Sx_{2n(i)}\right)$$
$$= d\left(Sx_{2m(i)}, Sx_{2n(i)-2}\right) + d\left(Sx_{2n(i)-2}, Sx_{2n(i)-1}\right)$$
$$+ d\left(Sx_{2n(i)-1}, Sx_{2n(i)}\right)$$
$$< \varepsilon + d\left(Sx_{2n(i)-2}, Sx_{2n(i)-1}\right) + d\left(Sx_{2n(i)-1}, Sx_{2n(i)}\right).$$

On letting $i \to \infty$ in above inequality and using (12), we have

$$\lim_{i\to\infty} d\left(Sx_{2m(i)}, Sx_{2n(i)}\right) = \varepsilon. \quad (17)$$

Also,

$$\varepsilon \le d\left(Sx_{2m(i)}, Sx_{2n(i)}\right)$$
$$\le d\left(Sx_{2m(i)}, Sx_{2m(i)-1}\right) + \varepsilon \le d\left(Sx_{2m(i)-1}, Sx_{2n(i)}\right)$$
$$\le 2d\left(Sx_{2m(i)}, Sx_{2m(i)-1}\right) + \varepsilon \le d\left(Sx_{2m(i)-1}, Sx_{2n(i)}\right).$$

Using (3),(8) and $i \to \infty$, we get

$$\begin{aligned}&\lim_{i\to\infty} d\left(Sx_{2m(i)}, Sx_{2n(i)}\right)\\ &= \lim_{i\to\infty} d\left(Sx_{2m(i)-1}, Sx_{2n(i)}\right) = \varepsilon.\end{aligned} \quad (18)$$

On the other hand, we have

$$d\left(Sx_{2m(i)}, Sx_{2n(i)}\right)$$
$$\le d\left(Sx_{2m(i)}, Sx_{2n(i)+1}\right) + d\left(Sx_{2n(i)+1}, Sx_{2n(i)}\right).$$

Letting $i \to \infty$, and using continuity of ψ, we have

$$\psi(\varepsilon) \le \lim_{i\to\infty} \psi\left(d\left(Sx_{2m(i)}, Sx_{2n(i)+1}\right)\right) \quad (19)$$

From (10)

$$\begin{aligned}&\psi\left(d\left(Sx_{2n(i)+1}, Sx_{2m(i)}\right)\right)\\ &\le \alpha\left(x_{2m(i)}, Tx_{2m(i)}\right)\beta\left(x_{2n(i)+1}, Tx_{2n(i)+1}\right)\\ &\quad\times \psi\left(d\left(Sx_{2n(i)+1}, Sx_{2m(i)}\right)\right)\\ &\le \psi\left(d\left(Tx_{2n(i)+1}, Tx_{2m(i)}\right)\right) - \varphi\left(d\left(Tx_{2n(i)+1}, Tx_{2m(i)}\right)\right)\\ &= \psi\left(d\left(Sx_{2n(i)}, Sx_{2m(i)-1}\right)\right) - \varphi\left(d\left(Sx_{2n(i)}, Sx_{2m(i)-1}\right)\right)\end{aligned} \quad (20)$$

From (18), $\lim_{i\to\infty} d\left(Sx_{2n(i)}, Sx_{2m(i)-1}\right) = \varepsilon.$

Again from (9), we have to prove that

$$\lim_{i\to\infty} d\left(Sx_{2n(i)+1}, Sx_{2m(i)-1}\right) = \varepsilon.$$

Using triangular inequality, we get

$$\left|d\left(Sx_{2n(i)+1}, Sx_{2m(i)-1}\right) - d\left(Sx_{2m(i)-1}, Sx_{2n(i)}\right)\right|$$
$$\le d\left(Sx_{2n(i)+1}, Sx_{2n(i)}\right).$$

Letting $i \to \infty$ in the above inequality and using (12) and (18), we obtain

$$\lim_{i\to\infty} d\left(Sx_{2n(i)+1}, Sx_{2m(i)-1}\right) = \varepsilon.$$

From (11), we have

$$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon) \Rightarrow \varphi(\varepsilon) = 0 \Rightarrow \varepsilon = 0,$$

a contradiction. Thus $\{Sx_{2n}\}$ is a Cauchy sequence in X, which gives that $\{Sx_n\}$ is a Cauchy sequence in X.

Step 3 Since by (11), we have $\{Sx_n\} = \{Tx_{n+1}\} \subseteq T(X)$ and $T(X)$ is closed, there exist $z \in X$ such that

$$\lim_{n\to\infty} Tx_n = Tz. \quad (21)$$

Now, we show that z is a coincidence point of S and T. On contrary, assume that $d(Sz, Tz) > 0$. Since by condition (iii) and (21), we have $\alpha\left(Tx_{n(k)}, Tz\right) \ge 1$ and $\beta\left(Tx_{n(k)}, Tz\right) \ge 1$ for all k, then by use of triangle inequality and (10)we have

$$d(Tz, Sz) \le d\left(Tz, Sx_{n(k)}\right) + d\left(Sx_{n(k)}, Sz\right)$$
$$\le d\left(Tz, Sx_{n(k)}\right) + \alpha(z, Tz)$$
$$\times \beta\left(x_{n(k)}, Tx_{n(k)}\right) d\left(Sx_{n(k)}, Sz\right)$$
$$\le d\left(Tz, Sx_{n(k)}\right) + \psi\left(d\left(Tx_{n(k)}, Tz\right)\right) - \varphi\left(d\left(Tx_{n(k)}, Tz\right)\right)$$

Letting $k \to \infty$, we get $d(Sz, Tz) \le 0$, a contradiction. Hence our supposition is wrong and $d(Sz, Tz) = 0$, that is $Sz = Tz$ .This shows that S and T have a coincidence point.

Definition: Let (X, d, ρ) be a digital metric space. Then the self maps $S, T : X \to X$ are called generalized $\alpha - \psi - \varphi$ -contractive type 1 mappings if there exist three functions $\alpha : X \times X \to [0, \infty)$ and $\psi, \varphi \in \phi$ such that

$$\begin{aligned}&\alpha(Tx, Ty)\psi(d(Sx, Sy))\\ &\le \psi(M(x, y)) - \varphi(M(x, y)), \forall x, y \in X,\end{aligned} \quad (22)$$

where

$$M(x, y) = \max\left\{\begin{matrix} d(Tx, Ty), d(Tx, Sx), \\ d(Sy, Ty), d(Sx, Ty) \end{matrix}\right\}.$$

Let $S, T : X \to X$ be two mappings. We denote by $C(S, T)$ the set of coincidence points of S and T; that is, $C(S, T) = \{z \in X : Sz = Tz\}$

Theorem 3.3: Let (X,d,ρ) be a complete digital metric space and $S,T:X\to X$ and $\alpha,\beta:X\times X\to[0,\infty)$ be mappings satisfying the following conditions :

i) $S(X)\subseteq T(X)$;

ii) S is $\alpha-\beta$-admissible w.r.t T;

iii) There exist $x_0\in X$ such that $\alpha(Tx_0,Sx_0)\geq 1$ and $\beta(Tx_0,Sx_0)\geq 1$

iv) $$\alpha(Tx,Ty)\psi(d(Sx,Sy)) \leq \psi(M(x,y))-\varphi(M(x,y)),\forall x,y\in X,$$

and $\psi,\varphi\in\phi$, where

$$M(x,y)=\max\begin{Bmatrix} d(Tx,Ty),d(Tx,Sx),\\ d(Sy,Ty),d(Sx,Ty)\end{Bmatrix}$$

v) If $\{Tx_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ and $\beta(x_n,x_{n+1})\geq 1$ for all n and $Tx_n\to Tx\in T(X)$ as $n\to\infty$, then there exists a subsequence $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ such that $\alpha(Tx_{n(k)},Tz)\geq 1$ and $\beta(Tx_{n(k)},Tz)\geq 1$ for all k.

Also suppose $T(X)$ is closed. Then S and T have a coincide point.

Proof: Proceeding as in theorem 3.2, we can have

$$\alpha(Tx_n,Tx_{n+1})\geq 1 \text{ and } \beta(Tx_n,Tx_{n+1})\geq 1,\ \forall n=0,1,2,\ldots \quad (23)$$

If $Sx_n=Sx_{n+1}$ for some n, then by (11), $Tx_{n+1}=Sx_{n+1}$, $n=0,1,2,\ldots$, that is S and T have a coincide point at $x=x_{n+1}$ and hence the proof. For this, we suppose that $d(Sx_n,Sx_{n+1})>0$ for all n. Applying (22)and (23),

$$\begin{aligned}&\psi(d(Sx_n,Sx_{n+1}))\\ &\leq \alpha(x_n,Tx_n)\beta(x_{n+1},Tx_{n+1})\psi(d(Sx_n,Sx_{n+1})) \quad (24)\\ &\leq \psi(M(x_n,x_{n+1}))-\varphi(M(x_n,x_{n+1})),\end{aligned}$$

where

$$\begin{aligned}&M(x_n,x_{n+1})\\ &=\max\begin{Bmatrix} d(Tx_n,Tx_{n+1}),d(Tx_n,Sx_n),\\ d(Sx_{n+1},Tx_{n+1}),d(Sx_n,Tx_{n+1})\end{Bmatrix}\\ &=\max\begin{Bmatrix} d(Sx_{n-1},Sx_n),d(Sx_{n-1},Sx_n),\\ d(Sx_{n+1},Sx_n),d(Sx_n,Sx_n)\end{Bmatrix}\\ &=\max\{d(Sx_{n-1},Sx_n),d(Sx_{n+1},Sx_n)\}.\end{aligned}$$

If for some $n\geq 1$, we have

$$d(Sx_{n-1},Sx_n)\leq d(Sx_{n+1},Sx_n),$$

then

$$\begin{aligned}&\psi(d(Sx_n,Sx_{n+1}))\\ &\leq \psi(d(Sx_{n+1},Sx_n))-\varphi(d(Sx_{n+1},Sx_n))\\ &\leq \psi(d(Sx_{n+1},Sx_n)),\end{aligned}$$

a contradiction and hence

$$\begin{aligned}&\psi(d(Sx_n,Sx_{n+1}))\\ &\leq \psi(d(Sx_{n-1},Sx_n))-\varphi(d(Sx_{n-1},Sx_n))\\ &\leq \psi(d(Sx_{n-1},Sx_n)).\end{aligned}$$

As ψ is non decreasing, we get

$$\{d(Sx_n,Sx_{n+1}):n\geq 0\}$$

is a non-increasing sequence. Hence there is $r\geq 0$ such that $\lim_{n\to\infty} d(Sx_n,Sx_{n+1})=r$. Letting $n\to\infty$ in (14), we have

$$\psi(r)\leq\psi(r)-\varphi(r)\Rightarrow\varphi(r)=0$$

and hence $r=0$.

Step 2: We will prove that $\{Sx_n\}$ is a Cauchy sequence. From (3), it is sufficient to show that $\{Sx_{2n}\}$ is a Cauchy sequence. Suppose to contrary; that is, $\{Sx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon>0$ for which we can find two subsequences of positive integers $(m(i))$ and $(n(i))$ such that $n(i)$ is smallest index for which

$$n(i)>m(i)>i, d(Sx_{2m(i)},Sx_{2n(i)})\geq\varepsilon. \quad (25)$$

This means that

$$d(Sx_{2m(i)},Sx_{2n(i)-2})<\varepsilon. \quad (26)$$

From (25)and (26)and triangular inequality, we get

$$\begin{aligned}&\varepsilon\leq d(Sx_{2m(i)},Sx_{2n(i)})\\ &= d(Sx_{2m(i)},Sx_{2n(i)-2})+d(Sx_{2n(i)-2},Sx_{2n(i)-1})\\ &\quad +d(Sx_{2n(i)-1},Sx_{2n(i)})\\ &<\varepsilon+d(Sx_{2n(i)-2},Sx_{2n(i)-1})+d(Sx_{2n(i)-1},Sx_{2n(i)}).\end{aligned}$$

On letting $i\to\infty$ in above inequality and using (12), we have

$$\lim_{i\to\infty} d(Sx_{2m(i)},Sx_{2n(i)})=\varepsilon. \quad (27)$$

Also,

$$\begin{aligned}&\varepsilon\leq d(Sx_{2m(i)},Sx_{2n(i)})\\ &\leq d(Sx_{2m(i)},Sx_{2m(i)-1})+\varepsilon\leq d(Sx_{2m(i)-1},Sx_{2n(i)})\\ &\leq 2d(Sx_{2m(i)},Sx_{2m(i)-1})+\varepsilon\leq d(Sx_{2m(i)-1},Sx_{2n(i)}).\end{aligned}$$

Using (12), (27)and $i\to\infty$, we get

$$\lim_{i\to\infty} d\left(Sx_{2m(i)}, Sx_{2n(i)}\right) = \lim_{i\to\infty} d\left(Sx_{2m(i)-1}, Sx_{2n(i)}\right) = \varepsilon. \tag{28}$$

On the other hand, we have

$$d\left(Sx_{2m(i)}, Sx_{2n(i)}\right) \le d\left(Sx_{2m(i)}, Sx_{2n(i)+1}\right) + d\left(Sx_{2n(i)+1}, Sx_{2n(i)}\right).$$

Letting $i\to\infty$, and using continuity of ψ, we have

$$\psi(\varepsilon) \le \lim_{i\to\infty} \psi\left(d\left(Sx_{2m(i)}, Sx_{2n(i)+1}\right)\right). \tag{29}$$

From (13)

$$\begin{aligned}
&\psi\left(d\left(Sx_{2n(i)+1}, Sx_{2m(i)}\right)\right)\\
&\le \alpha\left(x_{2m(i)}, Tx_{2m(i)}\right)\beta\left(x_{2n(i)+1}, Tx_{2n(i)+1}\right)\\
&\quad\times\psi\left(d\left(Sx_{2n(i)+1}, Sx_{2m(i)}\right)\right)\\
&\le \psi\left(M\left(x_{2n(i)+1}, x_{2m(i)}\right)\right) - \varphi\left(M\left(x_{2n(i)+1}, x_{2m(i)}\right)\right)
\end{aligned} \tag{30}$$

where,

$$\begin{aligned}
&M\left(x_{2m(i)}, x_{2n(i)+1}\right)\\
&= \max\left\{\begin{matrix} d\left(Tx_{2m(i)}, Tx_{2n(i)+1}\right), d\left(Tx_{2m(i)}, Sx_{2m(i)}\right),\\ d\left(Sx_{2n(i)+1}, Tx_{2n(i)+1}\right), d\left(Sx_{2m(i)}, Tx_{2n(i)+1}\right)\end{matrix}\right\}\\
&= \max\left\{\begin{matrix} d\left(Sx_{2m(i)-1}, Sx_{2n(i)}\right), d\left(Sx_{2m(i)-1}, Sx_{2m(i)}\right),\\ d\left(Sx_{2n(i)+1}, Sx_{2n(i)}\right), d\left(Sx_{2m(i)}, Sx_{2n(i)}\right)\end{matrix}\right\}.
\end{aligned}$$

From (19), $\lim_{i\to\infty} M\left(x_{2m(i)}, x_{2n(i)+1}\right) = \varepsilon.$

Also, using triangular inequality, we get

$$\left|d\left(Sx_{2n(i)+1}, Sx_{2m(i)-1}\right) - d\left(Sx_{2m(i)-1}, Sx_{2n(i)}\right)\right| \le d\left(Sx_{2n(i)+1}, Sx_{2n(i)}\right).$$

Letting $i\to\infty$ in the above inequality and using (12) and (28), we obtain

$$\lim_{i\to\infty} d\left(Sx_{2n(i)+1}, Sx_{2m(i)-1}\right) = \varepsilon.$$

From (21), we have

$$\psi(\varepsilon) \le \psi(\varepsilon) - \varphi(\varepsilon) \Rightarrow \varphi(\varepsilon) = 0 \Rightarrow \varepsilon = 0,$$

a contradiction. Thus $\{Sx_{2n}\}$ is a Cauchy sequence in X, which gives that $\{Sx_n\}$ is a Cauchy sequence in X.

Step 3 Since by (11), we have $\{Sx_n\} = \{Tx_{n+1}\} \subseteq T(X)$ and $T(X)$ is closed, there exist $z\in X$ such that $\lim_{n\to\infty} Tx_n = Tz.$

Now, we show that z is a coincidence point of S and T. On contrary, assume that $d(Sz, Tz) > 0$. Since by condition (v) and (21), we have $\alpha\left(Tx_{n(k)}, Tz\right) \ge 1$ and $\beta\left(Tx_{n(k)}, Tz\right) \ge 1$ for all k, then by use of triangle inequality and (1) we have

$$\begin{aligned}
&d(Tz, Sz) \le d\left(Tz, Sx_{n(k)}\right) + d\left(Sx_{n(k)}, Sz\right)\\
&\le d\left(Tz, Sx_{n(k)}\right)\\
&\quad+\alpha(z, Tz)\beta\left(x_{n(k)}, Tx_{n(k)}\right)d\left(Sx_{n(k)}, Sz\right)\\
&\le d\left(Tz, Sx_{n(k)}\right) + \psi\left(M\left(x_{n(k)}, z\right)\right) - \varphi\left(M\left(x_{n(k)}, z\right)\right)
\end{aligned} \tag{31}$$

Also,

$$M\left(x_{n(k)}, z\right) = \max\left\{\begin{matrix} d\left(Tx_{n(k)}, Tz\right), d\left(Tx_{n(k)}, Sx_{n(k)}\right),\\ d(Sz, Tz), d\left(Sx_{n(k)}, Tz\right)\end{matrix}\right\}.$$

Letting $k\to\infty$ in (22), we get $d(Sz, Tz) \le 0$, a contradiction. Hence our supposition is wrong and $d(Sz, Tz) = 0$, that is $Sz = Tz$. This shows that S and T have a coincidence point.

Theorem 3.4: Let (X, d, ρ) be a complete digital metric space and $S, T : X \to X$ and $\alpha, \beta : X\times X \to [0,\infty)$ be mappings satisfying the following conditions :

i) $S(X) \subseteq T(X)$;

ii) S is $\alpha-\beta$-admissible w.r.t T;

iii) There exist $x_0 \in X$ such that $\alpha(Tx_0, Sx_0) \ge 1$;

iv)

$$\begin{aligned}
&\alpha(x, Tx)\beta(y, Ty)\psi(d(Sx, Sy))\\
&\le \psi(M(x, y)) - \varphi(M(x, y)), \forall x, y\in X, \varphi, \psi \in \phi
\end{aligned}$$

and

$$M(x, y) = \max\left\{\begin{matrix} d(Sx, Sy), d(Tx, Sx), d(Sy, Ty),\\ \frac{1}{2}\left[d(Sx, Ty) + d(Sy, Tx)\right]\end{matrix}\right\}$$

v) If $\{Tx_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $Tx_n \to Tx \in T(X)$ as $n\to\infty$, then there exists a subsequence $\left\{Tx_{n(k)}\right\}$ of $\{Tx_n\}$ such that $\alpha\left(Tx_{n(k)}, Tz\right) \ge 1$ for all k.

Also suppose $T(X)$ is closed. Then S and T have a coincide point.

Proof: Follows directly from theorem 3.2.

The next theorem shows that under additional hypotheses we can deduce the existence and uniqueness of a common fixed point.

Theorem 3.5: In addition to the hypotheses of theorem 3.2, suppose that for all $x, y \in C(S,T)$, there exists $z \in X$ such that $\alpha(Tx,Tz) \geq 1$, $\beta(Tx,Tz) \geq 1$ and $\alpha(Ty,Tz) \geq 1$, $\beta(Ty,Tz) \geq 1$ and S,T commute at coincidence points. Then S and T have a unique common fixed point.

Proof: We will complete the result in three steps :

Step1 Uniqueness of coincidence point: $z_0 = z$.

Following as in theorem 3.2, we can have $\{Sz_n\}$, a Cauchy sequence in X such that

$$\{Sz_n\} = \{Tz_{n+1}\} \subseteq T(X),$$

for all $n \geq 0$ and. As $T(X)$ is closed, there exists $z \in X$ such that $\lim_{n\to\infty} Tz_n = Tz$, then existence of coincidence point is guaranteed. Now, we prove that S and T have a unique coincidence point i.e if $x, y \in C(S,T)$, then $Tx = Ty$.

As $x, y \in C(S,T)$ and $z \in X$,

$$\begin{aligned} &\alpha(Tx,Tz) \geq 1, \beta(Tx,Tz) \geq 1 \\ &\text{and } \alpha(Ty,Tz) \geq 1, \beta(Ty,Tz) \geq 1. \end{aligned} \tag{32}$$

Since S is $\alpha-\beta$-admissible w.r.t T, we have from (32)

$$\begin{aligned} &\alpha(Tx,Tz_n) \geq 1, \beta(Tx,Tz_n) \geq 1 \\ &\text{and } \alpha(Ty,Tz_n) \geq 1, \beta(Ty,Tz_n) \geq 1 \end{aligned} \tag{33}$$

for all $n \geq 0$. Let if possible $d(Tx,Tz) \neq 0$. Applying (1) and (24), we have

$$\begin{aligned} &\alpha(Tx,Ty)\psi(Sx,Sy) \leq \psi(d(Tx,Ty)) - \varphi(d(Tx,Ty)) \\ &\psi(d(Tx,Tz_n)) = \psi(d(Sx,Sz_{n-1})) \\ &\leq \alpha(Tx,Tz_n)\psi(d(Sx,Sz_n)) \\ &\leq \psi(d(Tx,Tz_{n-1})) - \varphi(d(Tx,Tz_{n-1})). \end{aligned} \tag{34}$$

Letting $n \to \infty$ in above inequality, we get

$$\psi(d(Tx,Tz)) \leq \psi(d(Tx,Tz)) - \varphi(d(Tx,Tz)),$$

a contradiction and hence $Tx = Tz$. Similarly we can show that $Ty = Tz$ that yields $Tx = Ty$.

Step 2: Existence of common fixed point:

Let $x \in C(S,T)$ i.e $Sx = Tx$. Owing to the commutativity of S and T at their coincidence points, we get

$$T^2x = TSx = STx \tag{35}$$

Let us denote $Tx = w$, then from (26), $Tw = Sw$. Thus w is a coincidence point of S and T. Now, from Step 1, we have $Tx = Tw = w = Sw$ and hence w is a common fixed point of S and T.

Step 3: Uniqueness: Assume that w' is another common fixed point of S and T. Then $w' \in C(S,T)$. By Step 1 we have $w' = Sw = Tw = w$. This completes the proof.

Example: Let $X = \{0,1,2,...\}, d(x,y) = |x-y|$ and $(X,d,1)$ be a digital metric space in Z with 1-adjacency. Define the self maps $S,T : X \to X$ by

$$S(x) = \begin{Bmatrix} 0, 0 \leq x \leq 5 \\ x-1, x > 5 \end{Bmatrix}, T(x) = x.$$

Also, define the mappings $\alpha, \beta : X \times X \to [0,\infty)$ as

$$\alpha(x,y) = \begin{Bmatrix} 1, x,y \leq 4 \\ 0, otherwise \end{Bmatrix} \text{ and } \beta(x,y) = \begin{Bmatrix} 1, x,y \leq 2 \\ \frac{3}{5}, otherwise \end{Bmatrix}.$$

Clearly, the pair (S,T) is $\alpha-\beta-\psi-\varphi$ contractive with $\psi(t) = \frac{3}{2}t$ and $\varphi(t) = \frac{1}{2}t$ for all $t \geq 0$. In fact for all $x, y \in X$, we have

$$\begin{aligned} &\alpha(x,Tx)\beta(y,Ty)d(Sx,Sy) = 1 \cdot 0 \\ &\leq \frac{3}{2}|x-y| - \frac{1}{2}|x-y| = |x-y| \end{aligned}$$

Moreover, there exist $x_0 \in X$ such that $\alpha(Sx_0,Tx_0) \geq 1$. In fact for $x_0 = 1$, we have

$$\alpha(Sx_0,Tx_0) = \alpha(S1,T1) = \alpha(0,1) \geq 1.$$

Now, we show that S is $\alpha-\beta$-admissible w.r.t T. For this, let $x, y \in X$, such that $\alpha(Tx,Ty) \geq 1 \Rightarrow Tx,Ty \leq 4$ i.e $x, y \leq 4$ and therefore $\alpha(Sx,Sy) = \alpha(0,0) = 1$ and

$$\beta(Tx,Ty) \geq 1 \Rightarrow Tx,Ty \leq 2$$

i.e $x, y \leq 2$ and therefore $\alpha(Sx,Sy) = \alpha(0,0) = 1$.

Also $S(X) \subseteq T(X)$ and $T(X)$ is closed. Lastly, let $\{Tx_n\}$ be a sequence in X such that

$$\alpha(Tx_n,Tx_{n+1}) \geq 1, \beta(Tx_n,Tx_{n+1}) \geq 1$$

for all n and $Tx_n \to Tz \in T(X)$ as $n \to \infty$ .Since $\alpha(Tx_n,Tx_{n+1}) \geq 1, \beta(Tx_n,Tx_{n+1}) \geq 1$, for all n, by definition of α and β we have $Tx_n \leq 2$ for all n and $Tz \leq 2$. Then $\alpha(Tx_n,Tz) \geq 1, \beta(Tx_n,Tz) \geq 1$ and hence all the conditions of theorem 3.5 are satisfied and consequently, $x = 0$ is common fixed point of S and T.

Remarks

Letting $S = I$ in theorem 3.1, we obtain Theorem 3.10 in [22]

Letting $S = I$ in theorem 3.2, we obtain Theorem 3.12 in [22]

Fixed Point Theorems for Cyclic Contractive Mappings. As a generalization of the Banach contraction mapping principle, Kirk et al. [22] in 2003 introduced cyclic representations and cyclic contractions. A mapping $T : A \cup B \to A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$, where A, B are nonempty subsets of a metric space (X,d). Moreover, T is called a cyclic contraction if there exists $\alpha \in (0,1)$ such that $d(Tx,Ty) \le \alpha d(x,y)$ for all $x \in A$ and $y \in B.$ Notice that although a contraction is continuous, cyclic contractions need not be. This is one of the important gains of this theorem. In the last decade, several authors have used the cyclic representations and cyclic contractions to obtain various fixed point results. see for example.

Theorem 4.1 Let (X,d,ρ) be complete digital metric space, A and B be two nonempty closed subsets of X. Suppose that $\alpha : X \times X \to [0,\infty)$ and $T : A \cup B \to A \cup B$ be the mappings such that $T(A) \subseteq B$ and $T(B) \subseteq A$, $\alpha(Tx,Ty) \ge 1$ when $\alpha(x,y) \ge 1$, satisfying the following conditions:

i) There exist $x_0 \in X$ such that $\alpha(x_0,Tx_0) \ge 1$

ii) Either T is continuous or X is α -regular

$$\alpha(x,y)\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \varphi(d(x,y)), \forall x,y \in X,$$

and $\psi,\varphi \in \phi$. Then T has a fixed point in $A \cap B.$ Further if u,v are fixed points of T with $\alpha(u,Tu) \ge 1, \alpha(v,Tv) \ge 1$ and $\beta(u,Tu) \ge 1, \beta(v,Tv) \ge 1,$ then $u = v.$

Proof: Let $Y = A \cup B$ and $\beta : Y \times Y \to [0,\infty)$ defined as $\beta(x,y) = \begin{Bmatrix} 1, x \in A, y \in B \\ 0, otherwise \end{Bmatrix}$, then (Y,d,ρ) be complete digital metric space. Now if $x_0 \in A$ such that $\alpha(x_0,Tx_0) \ge 1$ then also $\beta(x_0,Tx_0) \ge 1$ and hence all the hypotheses of theorem 3.1 are satisfied with $X = Y$, consequently, T has a fixed point in $A \cup B$, say z. If $z \in A$ implies $T(z) \in B$ and $z \in B$ implies $T(z) \in A,$ hence $z \in A \cap B$. Further if u,v are fixed points of $T,$ then $u,v \in A \cap B.$ Then $u \in A \Rightarrow T(u) \in B \Rightarrow \beta(u,Tu) \ge 1$ and $u \in B \Rightarrow T(u) \in A \Rightarrow \beta(u,Tu) \ge 1$, thus we can say $\beta(u,Tu) \ge 1, \beta(v,Tv) \ge 1.$ We deduce that all the conditions of theorem 3.1 are satisfied with $X = Y$ and hence T has a fixed point.

Theorem 4.2: Let (X,d,ρ) be complete digital metric space, A and B be two nonempty closed subsets of X and $S,T : Y \to Y$ be the mappings, where $Y = A \cup B$, satisfying the following conditions:

i) $T(A)$ and $T(B)$ are closed ;

ii) $S(A) \subseteq T(B)$ and $S(B) \subseteq T(A)$;

iii) T is one one ;

iv) There exist functions $\psi,\varphi \in \phi$ such that

$$\psi(d(Sx,Sy)) \le \psi(M(x,y)) - \varphi(M(x,y)), \forall x,y \in A \times B,$$

and

$$M(x,y) = \max \begin{Bmatrix} d(Tx,Ty), d(Tx,Sx), \\ d(Sy,Ty), d(Sx,Ty) \end{Bmatrix}.$$

Then S and T have a coincidence point in $A \cap B.$ Further, if S,T commute at their coincidence point , then S and T have a unique common fixed point in $A \cap B.$

Proof: Since A and B are closed subsets of $(X,d,\rho),$ so (Y,d,ρ) be complete digital metric space. Define the mappings $\alpha,\beta : Y \times Y \to [0,\infty)$ defined as

$$\alpha(x,y) = \beta(x,y) = \begin{Bmatrix} 1, (x,y) \in (T(A) \times T(B) \cup T(B) \times T(A)) \\ 0, otherwise \end{Bmatrix}.$$

Now if $x_0 \in Y = A \cup B,$ then we need to prove that $\alpha(Sx_0,Tx_0) \ge 1$ and $\beta(Sx_0,Tx_0) \ge 1.$ If $x_0 \in A,$ then $Sx_0 \in S(A) \subseteq T(B)$ and $Tx_0 \in T(A)$ and thus $(Sx_0,Tx_0) \in (T(A) \times T(B) \cup T(B) \times T(A))$, which proves $\alpha(Sx_0,Tx_0) \ge 1$ and $\beta(Sx_0,Tx_0) \ge 1.$

As T is one one, condition (iv) is equivalent to

$$\alpha(x,Tx)\beta(y,Ty)\psi(d(Sx,Sy)) \le \psi(M(x,y)) - \varphi(M(x,y)), \forall (x,y) \in T(A) \times T(B).$$

By using (ii), we can show that

From (ii), we have $S(Y) \subseteq T(Y).$ Moreover, is $T(Y)$ closed. Now, we proceed to show that S is α -admissible w.r.t S . Let

$$\alpha(Tx,Ty) \ge 1 \Rightarrow (Tx,Ty) \in (T(A) \times T(B) \cup T(B) \times T(A))$$

Since T is one one, we have

$$(x,y) \in ((A \times B) \cup (B \times A)),$$

thus

$$(Sx,Sy) \in (S(A) \times S(B) \cup S(B) \times S(A)) \subseteq (T(B) \times T(A) \cup T(A) \times T(B)),$$

which gives that $\alpha(Sx,Sy) \ge 1$. Similarly, we can show that $\beta(Tx,Ty) \ge 1$ implies $\beta(Sx,Sy) \ge 1.$

Now, let $\{Tx_n\}$ be a sequence in X such that $\alpha(Tx_n,Tx_{n+1}) \ge 1, \beta(Tx_n,Tx_{n+1}) \ge 1$ for all n and $Tx_n \to Tx \in T(X)$ as $n \to \infty$. From the definition of α and β, we have

$$(Tx_n,Tx_{n+1}) \in (T(A) \times T(B) \cup T(B) \times T(A)).$$

Since $T(A) \times T(B) \cup T(B) \times T(A)$ is closed set, thus

$$(Tx,Tx) \in (T(A) \times T(B) \cup T(B) \times T(A)) \Rightarrow Tx \in T(A) \cap T(B),$$

therefore, $\alpha(Tx_n,Tx) \ge 1, \beta(Tx_n,Tx) \ge 1$ for all n . Thus all the hypothesis of theorem 3.2 are satisfied. Hence, we deduce that S and T have a coincidence point $z \in A \cup B,$ that is $Sz = Tz$. If $z \in A,$ then $S(z) \in T(B).$ On the other hand $Sz = Tz \in T(A)$. Then we get $Tz \in T(A) \cap T(B)$, using one one property of T, we have $z \in A \cap B$. Similarly, $z \in B$, we have $z \in A \cap B$.

Notice that if x is a coincidence point of S and T, then $x \in A \cap B$. Finally, let $x, y \in C(S,T)$, that is $x, y \in A \cap B$, $Sx = Tx$ and $Sy = Ty$. From the above observation, we have $w = x \in A \cap B$ implies that $Tw \in T(A \cap B) = T(A) \cap T(B)$ due to the fact that T is one one, we get $\alpha(Tx, Tw) \geq 1, \alpha(Ty, Tw) \geq 1$ and $\beta(Tx, Tw) \geq 1, \beta(Ty, Tw) \geq 1$. Then our claims holds

Now, all the hypotheses of Theorem 3.3 are satisfied. So we deduce that $z \in A \cap B$ is the unique common fixed point of S and T. This completes the proof.

The following results are immediate consequences of above theorem.

Corollary 4.3: Let (X, d, ρ) be complete digital metric space, A and B be two nonempty closed subsets of X and $S, T: Y \to Y$ be the mappings, where $Y = A \cup B$, satisfying the following conditions:

i) $T(A)$ and $T(B)$ are closed ;

ii) $S(A) \subseteq T(B)$ and $S(B) \subseteq T(A)$;

ii) T is one one;

iv) There exist functions $\psi, \varphi \in \phi$ such that

$$\psi(d(Sx, Sy)) \leq \psi(M(x,y)) - \varphi(M(x,y)), \forall x, y \in A \times B,$$

and

$$M(x,y) = \max\left\{ \begin{array}{l} d(Sx,Sy), d(Tx,Sx), d(Sy,Ty), \\ \frac{1}{2}\left[d(Sx,Ty) + d(Sy,Tx)\right] \end{array} \right\}.$$

Then S and T have a coincidence point in $A \cap B$. Further, if S, T commute at their coincidence point, then S and T have a unique common fixed point in $A \cap B$.

Corollary 4.4: Let (X, d, ρ) be complete digital metric space, A and B be two nonempty closed subsets of X and $S, T: Y \to Y$ be the mappings, where $Y = A \cup B$, satisfying the following conditions:

i) $T(A)$ and $T(B)$ are closed;

ii) $S(A) \subseteq T(B)$ and $S(B) \subseteq T(A)$;

iii) T is one one ;

iv) There exist functions $\psi, \varphi \in \phi$ such that

$$\psi(d(Sx, Sy)) \leq \psi(d(Tx,Ty)) - \varphi(d(Tx,Ty)), \forall x, y \in A \times B.$$

Then S and T have a coincidence point in $A \cap B$. Further, if S, T commute at their coincidence point, then S and T have a unique common fixed point in $A \cap B$.

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