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The Translational Hull of a Left Restriction Semigroup

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Abstract In this paper, the translational hull of a left restriction semigroup is considered. We prove that the translational hull of a left restriction semigroup is still of the same type. This result extends the result of Guo and Shum on translational hulls of type A semigroups given in 2003.

Keywords: *translational hulls, left adequate semigroups, left restriction semigroups*

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1. Introduction

Let S be a semigroup. A mapping $\lambda(\rho)$ from S to itself is called a left (right) translation of S if we have $\lambda(ab) = (\lambda a)b\ \big((ab)\rho = a(b\rho)\big)$ for all $a, b \in S$. A left translation λ and a right translation ρ are called linked if $a(\lambda b) = (a\rho)b$ for all $a, b \in S$, in which case the pair (λ, ρ) is called a bitranslation of S. Denote by $\Lambda(S)$ $(I(S))$ the set of left (right) translations of S. It is easy to see that $\Lambda(S)$ and $I(S)$ are both semigroups under the composition of mappings. And it is also easy to check that $\Omega(S)$, the set of bitranslations of S, constitutes a subsemigroup of $\Lambda(S) \times I(S)$. We call the semigroup $\Omega(S)$ the translational hull of S. The concept of translational hull of semigroups and rings was first introduced by Petrich in 1970 (see [1]). The translational hull of an inverse semigroup was first studied by Ault [2] in 1973.

Later on, Fountain and Lawson [3] further studied the translational hulls of adequate semigroups. Guo and Shum [4] investigated the translational hull of type A semigroup, in particular, the result obtained by Ault [2] was substantially generalized and extended. Thus, the translational hull of a semigroup plays an important role in the theory of semigroups.

On the other hand, left restriction semigroups are class of semigroups which generalize inverse semigroups and which emerge very naturally from the study of partial transformation of a set. A more detailed description of left restriction semigroups can be found in [5] and [6].

Following Fountain [7], a semigroup S is said to be left abundant if each $\mathcal{R}^*$ - class of S contains at least one idempotent. Dually, right abundant semigroup can be defined. The semigroup S is called abundant if S is both left abundant and right abundant. As in [8], a left (right) abundant semigroup is called a left (right) adequate semigroup if the set of idempotents of S (i.e. $E(S)$) forms a semilattice. Regular semigroups are abundant semigroups and inverse semigroups are adequate semigroups.

In this paper, we shall show that the translational hull of a left restriction semigroup is still the same type. Thus, the result obtained by Guo and Shum in [4] for the translational hull of type A semigroup will be amplified.

2. Preliminaries

In this section we recall some definitions as well as some known results which will be useful in the sequel. We will use the notions and terminologies in [3], [4], [8], and [9].

Definition 2.1 [8]. Let S be a semigroup. Then S is said to be left (right) ample if

i) $E(S)$ is a semilattice.

ii) every element $a \in S$ is $\mathcal{R}^*(\mathcal{L}^*)-$ related to an idempotent, denoted by $a^\dagger$ (a^*).

iii) for all $a \in S$ and all $e \in E(S)$,

$$ae = (ae)^\dagger a (ea = a(ea)^*).$$

Definition 2.2 [3]. Let S be a semigroup and let $E \subseteq E(S)$ (E is the distinguished semilattice of idempotents).

Let $a, b \in S$, we have following relations on S

$$a\widetilde{\mathcal{R}}_E b \Leftrightarrow \forall\ e \in E,\quad ea = a \Leftrightarrow eb = b$$

$$a\widetilde{\mathcal{L}}_E b \Leftrightarrow \forall\ e \in E, ae = a \Leftrightarrow be = b.$$

Definition 2.3 [6]. Let S be a semigroup and let $E \subseteq E(S)$. Then S is said to be left (right) restriction semigroup if

i) E is a semilattice.

ii) every element $a \in S$ is $\widetilde{\mathcal{R}}_E\ (\widetilde{\mathcal{L}}_E)-$ related to an idempotent of E, denoted by $a^\dagger$ (a^*)

iii) the relation $\widetilde{\mathcal{R}}_E\ (\widetilde{\mathcal{L}}_E)$ is a left (right) congruence

iv) the left (right) ample condition holds:

$$ae = (ae)^\dagger a\ (ea = a(ea)^*).$$

The following Lemmas are due to Fountain [8] and Gould [6].

Lemma 2.4 [7]. Let S be a semigroup and e be an idempotent in S. Then the following are equivalent for $a \in S$.

i) $a\,\mathcal{R}^*e$
ii) $ea = a$, and for all $x, y \in S^1$, $xa = ya \Rightarrow xe = ye$.

Lemma 2.5 [6]. Let S be a semigroup and $E \subseteq E(S)$, let $a \in S, e \in E$. Then the following conditions are equivalent:
i) $a\,\tilde{\mathcal{R}}_E\,e$
ii) $ea = a$ and for all $f \in E, fa = a \Rightarrow fe = e$.

In a similar way to the *-relations, the ~ -relations are also related to the Green's relations as follows:

Lemma 2.6. In any semigroup S we have $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \tilde{\mathcal{R}}_E$. If S is regular, and $E = E(S)$ then $\tilde{\mathcal{R}}_E \subseteq \mathcal{R}$ and so $\tilde{\mathcal{R}}_E \subseteq \mathcal{R}^*$.

Dually we have $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \tilde{\mathcal{L}}_E$, and if S is regular, and $E = E(S)$ then $\tilde{\mathcal{L}}_E \subseteq \mathcal{L}$ and so $\tilde{\mathcal{L}}_E \subseteq \mathcal{L}^*$.

We note the following useful Lemma, the proof for which in [8] for left adequate semigroups can be easily adapted for left restriction semigroups.

Lemma 2.7. Let S be a left restriction semigroup and let $a, b \in S$. Then
i) $a\,\tilde{\mathcal{R}}_E\,b$ if and only if $a^\dagger = b^\dagger$
ii) $(ab)^\dagger = (ab^\dagger)^\dagger$ for all $a, b \in S$
iii) $(ea)^\dagger = ea^\dagger$ and $e \in E$.
iv) $a^\dagger a = a$
v) $(a^\dagger)^\dagger = a^\dagger$
vi) $a^\dagger b^\dagger a^\dagger = a^\dagger b^\dagger$
vii) $a^\dagger (ab)^\dagger = (ab)^\dagger$
viii) $(a^\dagger b^\dagger)^\dagger = a^\dagger b^\dagger$

Proof. Clearly, i) holds by definition. For ii), since $\tilde{\mathcal{R}}_E$ is a left congruence on *S*, we have $ab\,\tilde{\mathcal{R}}_E\,ab^\dagger$. Now, by Lemma 2.5, we have $(ab)^\dagger = (ab^\dagger)^\dagger$. Part iii) follows immediately from ii). iv) – viii) can be easily checked.

Lemma 2.8. Let S be a left restriction semigroup. Suppose that λ_1, λ_2 (ρ_1, ρ_2) are left (right) translations of *S* whose restriction to *E* are equal. Then $\lambda_1 = \lambda_2$ $(\rho_1 = \rho_2)$.

Proof. Let $a \in S$ and $e \in E$ such that $a\,\tilde{\mathcal{R}}_E\,e$. It is known from Lemma 2.5 that $ea = a$ and so

$$\lambda_1 a = \lambda_1(ea) = \lambda_1(e)a = \lambda_2(ea) = \lambda_2 a\ .$$

Consequently, $\lambda_1 = \lambda_2$. Similarly, it follows that $\rho_1 = \rho_2$.

Lemma 2.9. Let S be a left restriction semigroup. If $(\lambda_i, \rho_i) \in \Omega(S)$, for $i = 1,2$, then the following statements are equivalent:
i) $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$
ii) $\rho_1 = \rho_2$
iii) $\lambda_1 = \lambda_2$.

Proof. Note that i) $\Leftrightarrow$ ii) is the dual of i) $\Leftrightarrow$ iii) and that i) $\Longrightarrow$ ii) is trivial. We need to verify ii) $\Longrightarrow$ i).

Now suppose we let $\rho_1 = \rho_2$. To show ii) $\Longrightarrow$ i), it suffices to verify that $\lambda_1 = \lambda_2$. To see this, let $e \in E$, then $e\rho_1 = e\rho_2$ and we have that

$$\begin{aligned}\lambda_1 e &= (\lambda_1 e)^\dagger (\lambda_1 e) = \left[(\lambda_1 e)^\dagger \rho_1\right] e \\ &= \left[(\lambda_1 e)^\dagger \rho_2\right] e = (\lambda_1 e)^\dagger (\lambda_2 e) \\ &= (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger (\lambda_2 e).\end{aligned}$$

Now since $\tilde{\mathcal{R}}_E$ is a left congruence and $\lambda_2 e\,\tilde{\mathcal{R}}_E\,(\lambda_2 e)^\dagger$ by Lemma 2.5 (i), we have that

$$(\lambda_1 e)^\dagger \tilde{\mathcal{R}}_E \lambda_1 e = (\lambda_1 e)^\dagger (\lambda_2 e) \tilde{\mathcal{R}}_E\ (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger,$$

thereby, $(\lambda_1 e)^\dagger = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger$ since each $\tilde{\mathcal{R}}_E$ –class of a left restriction semigroup contains exactly one idempotent. Similarly, $(\lambda_2 e)^\dagger = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger$. Hence $(\lambda_1 e)^\dagger = (\lambda_2 e)^\dagger$.

Consequently,

$$\lambda_1 e = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger (\lambda_2 e) = (\lambda_2 e)^\dagger (\lambda_2 e) = \lambda_2 e,$$

and hence $\lambda_1 = \lambda_2$, as required.

3. The Translational Hull

Throughout this section, *S* will denote a left restriction semigroup with distinguished semilattice of idempotents *E*.

Now let *S* be a left restriction semigroup with distinguished semilattice *E* of idempotents. Let $(\lambda, \rho) \in \Omega(S)$ and define the mappings $\lambda^\dagger$ and $\rho^\dagger$ of *S* to itself as follows;

$$a\rho^\dagger = a\ (\lambda a^\dagger)^\dagger, ?\lambda^\dagger a = (\lambda a^\dagger)^\dagger a,$$

for all $a \in S$.

For the mappings $\lambda^\dagger$ and $\rho^\dagger$, we have the following Lemma.

Lemma 3.1. Let $(\lambda, \rho) \in \Omega(S)$. Then for all $e \in E$,
i) $e\rho^\dagger = \lambda^\dagger e$ and $e\rho^\dagger \in E$
ii) $\lambda^\dagger e = (\lambda e)^\dagger$

Proof. For all $e \in E$ and by the definition of the mappings above we have that

$$e\rho^\dagger = e(\lambda e)^\dagger = (\lambda e)^\dagger e = \lambda^\dagger e.$$

Also, the element $e\rho^\dagger$ is clearly an idempotent.

ii) Since $\tilde{\mathcal{R}}_E$ is a left congruence on *S*, and using Lemma 2.5, we have $\lambda^\dagger e = (\lambda e)^\dagger$, as required.

Lemma 3.2. The pair $(\lambda^\dagger, \rho^\dagger)$ is a member of the translational hull $\Omega(S)$ of *S*.

Proof. Suppose $a, b \in S$, using Lemma 2.7, we have

$\lambda^\dagger(ab) = [\lambda(ab)^\dagger]\,.ab$ (since $\lambda^\dagger a = (\lambda a^\dagger)^\dagger a$)
$= [\lambda(ab)^\dagger]^\dagger . a^\dagger . ab$ (by Lemma 2.7 (iv))
$= [\lambda(ab)^\dagger . a^\dagger]^\dagger\,.ab$
$= \{\lambda[(ab)^\dagger\, a^\dagger]\}^\dagger\,.ab$
(since $\lambda(ab) = (\lambda a)b$ where $a = (ab)^\dagger$ and $b = a^\dagger$)
$= \{\lambda[a^\dagger (ab)^\dagger\,]\}^\dagger\,.ab$
$= \{\lambda(ab)^\dagger\}^\dagger\,.ab$ (by Lemma 2.7 (vii))
$= \lambda^\dagger (ab)^\dagger . ab$ (by Lemma 2.7 (v))
$= (\lambda^\dagger a)b$ (by Lemma 2.7 (iv)).

We now prove that $\rho^\dagger$ is a right translation of *S*. For all $a, b \in S$, we first observe that $ab = (ab).b^\dagger$, by Lemma 2.7 (v), we have that $(ab)^\dagger = (ab)^\dagger b^\dagger$. So we have that

$(ab)^\dagger \rho^\dagger = ab.[\lambda(ab)^\dagger]^\dagger$
(since $a\rho^\dagger = a(\lambda a^\dagger)^\dagger$ where $a = ab$)
$= ab.\{\lambda[(ab)^\dagger\, b^\dagger]\}^\dagger$ (since $(ab)^\dagger = (ab)^\dagger b^\dagger$)
$= ab.\{\lambda[b^\dagger .(ab)^\dagger\,]\}^\dagger$
$= ab.[(\lambda b^\dagger).(ab)^\dagger]^\dagger$
(since $\lambda(ab) = (\lambda a)b$ where $a = b^\dagger$, $b = (ab)^\dagger$)
$= ab.[(\lambda b^\dagger).(ab)]^\dagger$ (by Lemma 2.7(ii))
$= ab.(\lambda b^\dagger)^\dagger.(ab)^\dagger$ (since $= (ab)^\dagger = a^\dagger b^\dagger$)
$= (ab)^\dagger . ab(\lambda b^\dagger)^\dagger$
$= a.b(\lambda b^\dagger)^\dagger$ (by Lemma 2.7(iv))
$= a(b\rho^\dagger)\,.$

So $\rho^\dagger$ is a right translation of S, as required.

To complete the proof, we proceed to show that the pair $(\lambda^\dagger, \rho^\dagger)$ are linked. We have that

$a(\lambda^\dagger b) = a.(\lambda b^\dagger)^\dagger b$ (since $\lambda^\dagger a = (\lambda a^\dagger)^\dagger a$)

$= a^\dagger . a . (\lambda b^\dagger)^\dagger . b$ (by Lemma 2.7(iv))

$= a.(\lambda b^\dagger)^\dagger a^\dagger . b$

$= a.[\lambda b^\dagger . a^\dagger]^\dagger . b$ (by Lemma 2.7(viii))

$= a.[\lambda(b^\dagger a^\dagger)]^\dagger . b$

(since $(\lambda a)b = \lambda(ab)$ where $a = b^\dagger$, $b = a^\dagger$)

$= a.[\lambda(a^\dagger b^\dagger)]^\dagger . b$

$= a.[\lambda a^\dagger . b^\dagger]^\dagger . b$

(since $\lambda(ab) = (\lambda a)b$ where $a = a^\dagger$, $b = b^\dagger$)

$= a.(\lambda a^\dagger)^\dagger . b^\dagger . b$ (by Lemma 2.7(viii))

$= a(\lambda a^\dagger)^\dagger b$ (by Lemma 2.7(iv))

$= (a\rho^\dagger)b$ (since $a\rho^\dagger = a(\lambda a^\dagger)^\dagger$)

Consequently, $(\lambda^\dagger, \rho^\dagger) \in \Omega(S)$.

Lemma 3.3. Suppose $\Gamma(S) = \{(\lambda, \rho) \in \Omega(S) : \lambda E \cup E\rho \subseteq E\}$. Then $\Gamma(S)$ is the distinguished semilattice of idempotents of $\Omega(S)$.

Proof. Suppose $(\lambda, \rho) \in \Omega(S)$ and $e \in E$. Then, $\lambda e \in E$ and $e\rho \in E$. Thus, we have

$$e\rho^2 = (e\rho)\rho = (e(e\rho))\rho = ((e\rho)e)\rho$$
$$= (e\rho)(e\rho) = e\rho,$$

and by Lemma 2.8, $\rho^2 = \rho$. Similarly, $\lambda^2 e = \lambda e$. By Lemma 2.9, it follows that $(\lambda, \rho)^2 = (\lambda, \rho)$.

Conversely, suppose that $\lambda^2 = \lambda$ and $\rho^2 = \rho$, then for all $e \in E$,

$$\lambda e = \lambda^2 e = \lambda(\lambda ee) = (\lambda e)^2 \subseteq E.$$

Similarly, it follows that $e\rho \subseteq E$. Consequently, $\lambda e \cup e\rho \subseteq E$ so that $(\lambda, \rho) \in \Gamma(S)$.

An immediate consequence of Lemma 3.1 – 3.3 is the following

Corollary 3.4. Let S be a left restriction semigroup and $(\lambda, \rho) \in \Omega(S)$. Then $(\lambda^\dagger, \rho^\dagger) \in E(\Omega(S))$.

Lemma 3.5. The elements (λ_1, ρ_1) and (λ_2, ρ_2) of $\Gamma(S)$ commute with each other.

Proof. For (λ_1, ρ_1), $(\lambda_2, \rho_2) \in \Gamma(S)$ and $e \in E$ we have that

$$\lambda_1 \lambda_2 e = \lambda_1 (\lambda_2 e) = \lambda_1 ((\lambda_2 e)e)$$
$$= (\lambda_1 e)(\lambda_2 e) = (\lambda_2 e)(\lambda_1 e) = \lambda_2 ((\lambda_1 e)e)$$
$$= \lambda_2 \lambda_1 e.$$

It follows from Lemma 2.8 that $\lambda_1\lambda_2 = \lambda_2\lambda_1$. Similarly, $e\rho_1\rho_2 = e\rho_2\rho_1$.

Consequently, It follows from Lemma 2.9 that $(\lambda_1\lambda_2, \rho_1\rho_2) = (\lambda_2\lambda_1, \rho_2\rho_1)$, that is we have that $(\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_2, \rho_2)(\lambda_1, \rho_1)$, as required.

Lemma 3.6. Let $(\lambda, \rho) \in \Omega(S)$. Then $(\lambda, \rho)(\lambda^\dagger, \rho^\dagger) = (\lambda, \rho) = (\lambda^\dagger, \rho^\dagger)(\lambda, \rho)$

Proof. For all $e \in E$, since

$\lambda \lambda^\dagger e = \lambda[(\lambda e^\dagger)^\dagger e]$ (since $\lambda^\dagger a = (\lambda e^\dagger)^\dagger e$)

$= \lambda[(\lambda e)^\dagger e]$

$= \lambda[e(\lambda e)^\dagger]$

$= \lambda e(\lambda e)^\dagger$

$= (\lambda e)^\dagger \lambda e = \lambda e,$

we have that $\lambda\lambda^\dagger = \lambda$, and by Lemma 2.9, $\rho = \rho\rho^\dagger$. This shows that $(\lambda, \rho)(\lambda^\dagger, \rho^\dagger) = (\lambda, \rho)$.

Similarly, it follows that $(\lambda^\dagger, \rho^\dagger)(\lambda, \rho) = (\lambda, \rho)$.

Lemma 3.7. Let S be a left restriction semigroup and $(\lambda, \rho) \in \Omega(S)$. Then $(\lambda, \rho)\ \tilde{\mathcal{R}}_{E(\Omega(S))}\ (\lambda^\dagger, \rho^\dagger)$.

Proof. Let $(\lambda^\dagger, \rho^\dagger)$ be an idempotent of $\Omega(S)$. That $(\lambda, \rho)\ \tilde{\mathcal{R}}_{E(\Omega(S))}\ (\lambda^\dagger, \rho^\dagger)$ entails showing that

$$(\lambda^\dagger, \rho^\dagger)(\lambda, \rho) = (\lambda, \rho) \Leftrightarrow (\lambda^\dagger, \rho^\dagger)(\lambda^\dagger, \rho^\dagger) = (\lambda^\dagger, \rho^\dagger).$$

That is, $(\lambda^\dagger\lambda, \rho^\dagger\rho) = (\lambda, \rho) \Leftrightarrow (\lambda^{+2}, \rho^{+2}) = (\lambda^\dagger, \rho^\dagger)$.

By Lemma 2.9, it entails showing that

$$\rho^\dagger\rho = \rho \Leftrightarrow \rho^{+2} = \rho^\dagger, \lambda^\dagger\lambda = \lambda \Leftrightarrow \lambda^{+2} = \lambda^\dagger.$$

Now suppose that $\lambda^\dagger\lambda = \lambda$. Then employ Lemma 2.7 to obtain the following

$(\lambda^\dagger\lambda)^\dagger = (\lambda^\dagger)^\dagger\lambda^\dagger$ (by Lemma 2.7 (viii))

$= \lambda^\dagger\lambda^\dagger$ (by Lemma 2.7 (vi))

$\Rightarrow \lambda^{+2} = \lambda^\dagger$.

It follows similarly for $\rho^\dagger\rho = \rho$.

Conversely, let $\lambda^{+2} = \lambda^\dagger$. Multiplying both sides by λ, we immediately have

$\lambda^\dagger\lambda^\dagger\lambda = \lambda^\dagger\lambda$

$\Rightarrow \lambda^\dagger\lambda = \lambda$ (by Lemma 2.7 (iv)).

It follows similarly for $\rho^\dagger\rho = \rho$.

Consequently, it can be easily seen that $(\lambda, \rho)\ \tilde{\mathcal{R}}_{E(\Omega(S))}\ (\lambda^\dagger, \rho^\dagger)$.

Lemma 3.8. $\tilde{\mathcal{R}}_{E(\Omega(S))}$ is a left congruence on $\Omega(S)$ for a left restriction semigroup S.

Proof. To show that $\tilde{\mathcal{R}}_{E(\Omega(S))}$ is a left congruence, let $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)$. Then

$$(\lambda_1, \rho_1)\tilde{\mathcal{R}}_{E(\Omega(S))}(\lambda_2, \rho_2) \Leftrightarrow ((\lambda_1, \rho_1))^\dagger = ((\lambda_2, \rho_2))^\dagger$$
$$\Leftrightarrow (\lambda_1^\dagger, \rho_1^\dagger) = (\lambda_2^\dagger, \rho_2^\dagger)$$
$$\Leftrightarrow \lambda_1^\dagger = \lambda_2^\dagger (\rho_1^\dagger = \rho_2^\dagger).$$

So we have that

$(\lambda_1, \rho_1)\ \tilde{\mathcal{R}}_{E(\Omega(S))}\ (\lambda_2, \rho_2) \Rightarrow \lambda_1^\dagger = \lambda_2^\dagger$

$\Rightarrow \lambda' \lambda_1^\dagger = \lambda' \lambda_2^\dagger$

$\Rightarrow (\lambda'\lambda_1, \rho'\rho_1) = (\lambda'\lambda_2, \rho'\rho_2)$

$\Rightarrow ((\lambda', \rho')(\lambda_1, \rho_1))^\dagger = ((\lambda', \rho')(\lambda_2, \rho_2))^\dagger$

$\Rightarrow (\lambda', \rho')(\lambda_1, \rho_1)\ \tilde{\mathcal{R}}_{E(\Omega(S))}\ (\lambda', \rho')(\lambda_2, \rho_2)$

for any $(\lambda', \rho') \in \in \Omega(S)$. Thus $\tilde{\mathcal{R}}_{E(\Omega(S))}$ is a left congruence.

Lemma 3.9. Let $(\lambda, \rho) \in \Omega(S)$. Then $(\lambda, \rho)(\lambda^\dagger, \rho^\dagger) = ((\lambda, \rho)(\lambda^\dagger, \rho^\dagger))^\dagger (\lambda, \rho)$.

Proof. From Lemma 3.6, we know that $(\lambda, \rho)(\lambda^\dagger, \rho^\dagger) = (\lambda, \rho)$.

Now, $((\lambda, \rho)(\lambda^\dagger, \rho^\dagger))^\dagger (\lambda, \rho) = (\lambda, \rho)^\dagger(\lambda, \rho) = (\lambda, \rho)$.

Consequently,

$$(\lambda, \rho)(\lambda^\dagger, \rho^\dagger) = (\lambda, \rho) = ((\lambda, \rho)(\lambda^\dagger, \rho^\dagger))^\dagger (\lambda, \rho).$$

Thus, $\Omega(S)$ is a left type A (since the left ample condition holds).

By using the above Lemmas 3.2 – 3.3, Corollary 3.4, Lemmas 3.5 – 3.9, we can easily verify that for any $(\lambda, \rho) \in \Omega(S)$ there exists a unique idempotent $(\lambda^\dagger, \rho^\dagger)$ such that $(\lambda, \rho)\ \tilde{\mathcal{R}}_{E(\Omega(S))}\ (\lambda^\dagger, \rho^\dagger)$ and $(\lambda^\dagger, \rho^\dagger)(\lambda, \rho) = (\lambda, \rho)$. Hence, $\Omega(S)$ is indeed a left restriction semigroup.

So far we have proved the following theorem:

Theorem 3.10. The translational hull of a left restriction semigroup is still a left restriction semigroup.

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