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On Weak Solutions of Systems of Strongly Nonlinear Parabolic Variational Inequalities

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Abstract In this paper we prove the existence of weak solutions for systems of variational inequalities of strongly nonlinear parabolic operators: $u_t^{(\ell)} + A^{(\ell)}(u)(x,t) + g^{(\ell)}\left(x,t;u^1,\ldots u^d\right)$, **in** $\mathbf{Q} = \Omega \times (\mathbf{o},\mathbf{T})$, where $A^{(\ell)}(u)(x,t) = \sum_{|\alpha|\le \mathbf{m}} (-1)^{|\alpha|} \mathbf{D}^{\alpha} \mathbf{A}_{\alpha}^{(\ell)}\left(x,t;D\left(u^1(x,t),\ldots u^d(x,t)\right)\right)$, $\ell = 1,2,\ldots,d$.

Keywords: *strongly nonlinear parabolic operators-Systems of variational inequalities*

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1. Introduction

Consider the parabolic initial-boundary value problem

$$u_t + A(u)(x,t) + g(x,t;u) = f(x,t), \qquad (1)$$

$$\textbf{in } \mathbf{Q} = \Omega \times (\mathbf{o},\mathbf{T})$$

$$u(0) = 0,$$

in Ω

$$\mathbf{D}^{\alpha}\mathbf{u} = 0 \textbf{ on } \partial\Omega \times (0,\mathbf{T}) \textbf{ for } |\alpha| \le \mathbf{m} - 1$$

where

$$\mathbf{A}(\mathbf{u})(\mathbf{x},\mathbf{t}) = \sum_{|\alpha|\le \mathbf{m}} (-1)^{|\alpha|} \mathbf{D}^{\alpha} \mathbf{A}_{\alpha}\left(x,t;Du(x,t)\right)$$

and Du= $(\mathbf{D}^{\alpha}\mathbf{u})_{|\alpha|\le m}$ .If the coefficients $\mathbf{A}_{\alpha}$ satisfy a polynomial growth conditions of order (p-1) in u and its space derivatives but g obeys no growth in u, but merely a sign condition, the existence of weak solutions problems of the type (1) has been obtained by many authors (cf [1,4] and [5]). In [2] Browder and Brézis extended the above results to the corresponding class of variational inequalities. Their proof based on a type of compactness result. Our result can be viewed as a generalization to systems of variational inequalities for the work of [4] and [5].Our proof relies on deriving a-priori bound for the time derivative of the solution in $\mathbf{L}^2(\mathbf{Q})$.

2. Prerequisites

Let Ω be a bounded domain in $\mathbb{R}^{\mathbf{N}}$ with a smooth (uniform C^m-) boundary, $1<p<\infty$ and m a positive integer. Denote by $V = \boldsymbol{V}^{(1)} \times \ldots \times \boldsymbol{V}^{(d)}$ the Sobolev space

$$\prod_{\ell=1}^{\mathbf{d}} W_0^{m,p}(\Omega) = [W_0^{m,p}(\Omega)]^d = \overline{[\mathbf{C}_0^{\infty}(\Omega)]^d}^{\| \|_{m,p}}$$

where

$$\|u\|^p_{m,p} = \int_{\Omega} \sum_{\ell=1}^{d} \sum_{|\alpha|\le \mathbf{m}} \mathbf{D}^{\alpha}\mathbf{u}^{(\ell)}(\mathbf{x})^p{}_p\, dx.$$

Let $X=L^p(0,T,V)$, $\mathbf{X}^* = L^{p'}(0,T,\mathbf{V}^*)$ be its topological dual ($p'=p/p-1$) and $Y=\{u: u\in X, \frac{du}{dt} \in \mathbf{X}^*, \boldsymbol{u}(\mathbf{0}) = \mathbf{0}\}$ with the norm

$$\|u\|_Y = \|u\|_X + \left\|\frac{du}{dt}\right\|_{\mathbf{X}^*}.$$

For the Galerkin method, construct a sequence $(\boldsymbol{w}_i^{(\ell)})_{i=1}^{\infty} \subset [\mathbf{C}_0^{\infty}(\Omega)]^{(\ell)}$ such that $\prod_{\ell=1}^{\mathbf{d}} \cup_{\mathbf{n}=1}^{\infty} \boldsymbol{Z}_n^{(\ell)}$ with $\boldsymbol{Z}_n^{(\ell)} = \boldsymbol{span}\left(\boldsymbol{w}_1^{(\ell)}, \ldots, \boldsymbol{w}_n^{(\ell)}\right)$ is dense in $[\boldsymbol{W}_0^{j,p}(\Omega)]^{d}$, $jp > mp + N$. Denote by $\mathbf{Y_n} = \mathbf{C}(0,\mathbf{T};Z_n^{(1)},\ldots.Z_n^{(\mathbf{d})})$. Since $[\boldsymbol{W}_0^{j,p}(\Omega)]^{d}$ is continuously embedded in $[\boldsymbol{C}_B^{m}(\Omega)]^{d}$, which is a Banach space with the norm

$$\|u\|_{[C_B^m(\Omega)]^d} = \sum_{\ell=1}^{d} \max_{0\le|\alpha|\le \mathbf{m}} \sup_{\mathbf{x}\in\Omega} \left|\mathbf{D}^{\alpha}\mathbf{u}^{(\ell)}(\mathbf{x})\right|,$$

then for any $v \in [\boldsymbol{W}_0^{j,p}(\Omega)]^{d}$ there exists a sequence $(v_k) \in \cup_{n=1}^{\infty} \mathbf{Y_n}$ such that $v_k \to \boldsymbol{v} \textbf{ in } [\boldsymbol{W}_0^{j,p}(\Omega)]^{d}$. Moreover, since the closure of $\cup_{n=1}^{\infty} \mathbf{Y_n}$ with respect to the $[C^m]^d$-topology contains $[\mathbf{C}_0^{\infty}(\mathbf{Q})]^{d}$, then for any $\boldsymbol{f} \in \mathbf{X}^*$ there exists a sequence (f_k) $\mathbf{f}_{\mathbf{k}}^{(\ell)} \in \cup_{\mathbf{n}=1}^{\infty} \boldsymbol{Z}_n^{(\ell)}$ such that $\mathbf{f_k} \to \mathbf{f}$ in $\mathbf{X}^*$ in the weak sense [5].

We introduce the following hypotheses for $\boldsymbol{A}^{(\ell)}(u) + \boldsymbol{g}^{(\ell)}(x,t;u)$.

A1) $\mathbf{A}_{\alpha}^{(\ell)}: \mathbf{Q} \times \mathbb{R}^{sd} \to \mathbb{R}$ is continuous in $\xi \in \mathbb{R}^{sd}$ for almost all $(x, t) \in \mathbf{Q}$ and $\mathbf{c_1} > \mathbf{0}$ and a fixed function $\mathbf{K_1} \epsilon [\mathbf{L}^{p'}(\mathbf{Q})]^{\mathbf{d}}$

$$\left|\mathbf{A}_{\alpha}^{(\ell)}(\mathbf{x},\mathbf{t};\xi)\right| \le \mathbf{c}_1|\xi|^{p-1} + \mathbf{K}_1(x,t),$$

for all α ,all (x,t) $\in \mathbf{Q}$, $\ell = 1, 2, \dots, \mathbf{d}$ *and all* $\xi \in \mathbb{R}^{sd}$.

A2) For all (x,t) $\in \mathbf{Q}$ and two distinct $\xi, \xi^* \in \mathbb{R}^{sd}$

$$\sum_{\ell=1}^{d}\sum_{|\alpha|\le \mathbf{m}}(\mathbf{A}_{\alpha}^{(\ell)}(\mathbf{x},\mathbf{t};\xi) - \mathbf{A}_{\alpha}^{(\ell)}(\mathbf{x},\mathbf{t};\xi^*))(\xi_{\alpha}^{(\ell)} - \xi_{\alpha}^{*(\ell)}) > 0.$$

A3) There exists a constant $c_2>0$ and a fixed function $\mathbf{K_2} \in \mathbf{L}^{p'}(\mathbf{Q})]^{\mathbf{d}}$ such that for all (x,t) $\in \mathbf{Q}$ and all $\xi \in \mathbb{R}^{sd}$

$$\sum_{\ell=1}^{d}\sum_{|\alpha|\le \mathbf{m}}\mathbf{A}_{\alpha}^{(\ell)}(\mathbf{x},\mathbf{t};\xi)\xi_{\alpha}^{(\ell)} \ge c_2|\xi|^p - \mathbf{K}_2(\mathbf{x},\mathbf{t}).$$

G) $\boldsymbol{g}^{(\ell)}: \mathbf{Q} \times \mathbb{R}^{d} \to \mathbb{R}$ is continuous in $\mathbf{r} \in \mathbb{R}^{d}$ for almost all (x,t) $\in \mathbf{Q}$ and measurable in $(\mathbf{x},\mathbf{t})$ for all $\mathbf{r}, \ell = 1, 2, \dots, \mathbf{d}$. Moreover, each $\boldsymbol{g}^{(\ell)}$ is nondecreasing in *r* for fixed $(\mathbf{x},\mathbf{t}) \in \mathbf{Q}$ and each $\boldsymbol{g}^{(\ell)}(\mathbf{x},\mathbf{t};\mathbf{0}) = \mathbf{0}$, for all $(\mathbf{x},\mathbf{t}) \in \mathbf{Q}$.

3. Formulation of the Problem

Write

$$G_0^{(\ell)}(x,t,r) = \int_0^r g^{(\ell)}(x,t;s)ds.$$

By G) each $\boldsymbol{G}_{\mathbf{0}}^{(\ell)}$ as a function of r is convex, nonnegative and once differentiable.

For u $\in \boldsymbol{X}$, **set**

$$\Gamma^{(\ell)}(u) = \int_Q G_0^{(\ell)}(\mathbf{x},\mathbf{t};\mathbf{u})dxdt.$$

Let K be a closed convex subset of V containing the origin. Define a proper lower semicontinuous Gateaux differentiable function $\boldsymbol{\varphi}$: X→ $(-\infty, \infty]$:

$$\varphi^{(\ell)}(u) = \begin{cases} \Gamma^{(\ell)}(\mathbf{u}) & \textbf{if } \mathbf{u}(\mathbf{t}) \in \mathbf{K} \textbf{ a.e} \\ \infty & \textit{otherwise.} \end{cases}$$

Definition: A function $u_n = (\boldsymbol{u}_{\boldsymbol{n}}^{(1)}, \dots \boldsymbol{u}_{\boldsymbol{n}}^{(d)}) \in \mathbf{Y}_n$ is called a Galerkin solution of the associated variational inequalities for (1) if

$$\begin{aligned} &\int_0^T\left(\frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}}{\partial \mathbf{t}}, \mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{n}}^{(\ell)}\right)\mathbf{dt} + \int_0^T\left(\mathbf{T}(u_n), \mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{n}}^{(\ell)}\right)\mathbf{dt} \\ &+\varphi^{(\ell)}(\mathbf{v}) - \varphi^{(\ell)}(\mathbf{u_n}) \\ &\ge \int_0^T(\mathbf{f}_{\mathbf{n}}^{(\ell)}, \mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{n}}^{(\ell)})\mathbf{dt}, \\ &v \in Y_{\mathbf{n}},\ \ell = 1,2,\dots,\mathbf{d}. \end{aligned} \qquad (2)$$

where

$$\left(\mathbf{T}(\mathbf{u}), \mathbf{z}^{(\ell)}\right) = \int_\Omega\sum_{\ell=1}^{d}\sum_{|\alpha|\le m}\mathbf{A}_{\alpha}^{(\ell)}(\mathbf{x},\mathbf{t};\mathbf{Du})D^\alpha \mathbf{z}^{(\ell)}dx.$$

The existence of a Galerkin solution and its main property, in view of our hypotheses,will be given by the following lemma [3].

Lemma: For every n∈ℕ there exists a Galerkin solution $u_n \in \mathbf{Y}_n \cap \boldsymbol{X}$ such that

$$\|\mathbf{u_n}\|_{2;2} \le c.\ (\mathbf{c} > \mathbf{0})$$

Proof: Define a vector-valued function h_n: $[\mathbf{o},\mathbf{T}] \times \mathbb{R}^s \to \mathbb{R}^s$ by

$$(\mathbf{h_n}(\mathbf{t},\mathbf{a}))_i = \int_\Omega\int_{\ell=1}^{d}\sum_{|\alpha|\le m}\mathbf{A}_{\alpha}^{(\ell)}(\sum_{\mathbf{j}=1}^{\mathbf{s}}\mathbf{a_j}\psi_j)D^\alpha\psi_i^{(\ell)}dx,$$

$$\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a_s}),\ \mathbf{i} = 1,\dots,\mathbf{n}.$$

$\mathbf{h_n}(\mathbf{t},\mathbf{a})$, in view of our hypotheses, is continuous and coercive an each $\boldsymbol{\varphi}^{(\ell)}: \mathbf{Y_n} \to (-\infty, \infty]$ is lower semicontinuous with $\boldsymbol{\varphi}^{(\ell)}(\mathbf{0}) = \mathbf{0}$.

Therefore the system

$$\begin{aligned} &(\dot{\xi}^{(\ell)}(t) + \mathbf{h_n}(\mathbf{t}, \xi(t)), \eta^{(\ell)} - \xi^{(\ell)}(\mathbf{t})) \\ &+\varphi^{(\ell)}(\eta) - \varphi^{(\ell)}(\xi(\mathbf{t})) \\ &\ge (\mathbf{f}_{\mathbf{n}}^{(\ell)}, \eta^{(\ell)} - \xi^{(\ell)}(\mathbf{t})) \end{aligned} \qquad (3)$$

for a.a.t∈ $[\mathbf{0},\mathbf{T}]$, $\boldsymbol{\xi}^{(\ell)}(\mathbf{0}) = \mathbf{0}$ with

$$(\mathbf{f}_{\mathbf{n}}^{(\ell)}(\mathbf{t}))_i = \int_\Omega \mathbf{f}_{\mathbf{n}}^{(\ell)}(\mathbf{t})\psi_i^{(\ell)}\mathbf{dx},\ \mathbf{i} = 1,\dots,\mathbf{n},$$

has a local solution.

From (3) we get the *e*stimate

$$\frac{1}{2}\frac{d}{dt}\left|\xi^{(\ell)}(\mathbf{t})\right| \le \left|\mathbf{f}_{\mathbf{n}}^{(\ell)}(\mathbf{t})\right|\left|\xi^{(\ell)}(\mathbf{t})\right|.$$

Therefore

$$\left|\xi^{(\ell)}(\mathbf{t})\right| \le c_n(T).$$

Including the existence of a solution

$$\xi(t) = (\xi^{(1)}(t), \dots, \xi^{(d)}(t)) \in \mathbf{Y_n}.$$

4. Existence Theorem

Theorem. Let the hypotheses A1)- A3) and G) be satisfied. Let $f^{(\ell)} \in c^1(0, T, L^2(\Omega))$ be given. Then

(i) there exists u∈ Y with u(t)∈ K a. e., u(0) = 0 such that

$$\begin{aligned} &< u_t^{(\ell)}, \mathbf{v}^{(\ell)} - \mathbf{u}^{(\ell)} > + < \mathbf{T}(u), \mathbf{v}^{(\ell)} - \mathbf{u}^{(\ell)} > \\ &+\varphi^{(\ell)}(\mathbf{v}) - \varphi^{(\ell)}(\mathbf{u_n}) \ge < \mathbf{f}^{(\ell)}, \mathbf{v}^{(\ell)} - \mathbf{u}^{(\ell)} >, \end{aligned}$$

for every v $\in \boldsymbol{C}^{\mathbf{1}}(\mathbf{0}, \boldsymbol{T}, [\mathbf{C}_{\mathbf{0}}^{\infty}(\Omega)]^{\mathbf{d}})$ for which $\boldsymbol{\varphi}^{(\ell)}(\mathbf{v}) < \infty$.

(ii) there exists u∈ $\mathbf{Y} \cap \mathbf{K}$ with u(t)∈ K a. e., u(0) = 0 such that

$$\begin{aligned} &< u_t^{(\ell)}, \mathbf{v}^{(\ell)} - \mathbf{u}^{(\ell)} > + < \mathbf{T}(u), \mathbf{v}^{(\ell)} - \mathbf{u}^{(\ell)} \\ &+\int_Q g^{(\ell)}(x,t,u)(\mathbf{v}^{(\ell)} - \mathbf{u}^{(\ell)})dxdt \\ &\ge < \mathbf{f}^{(\ell)}, \mathbf{v}^{(\ell)} - \mathbf{u}^{(\ell)} >, \end{aligned}$$

for every $v \in C^1(0,T,[C_0^\infty(\Omega)]^d)$ with $u(t) \in K$ a.e.

Proof of (i): By the above lemma, there exist Galerkin solutions $u_n \in Y_n$ of (2) such that

$$\|u_n(t)\|_2 \le c.$$

Set v=0 in (2) we get the uniform boundedness from above of the numerical sequence $\{(T(u_n), u_n^{(\ell)})\}_{n\in\mathbb{N}}$. The proof will follow if we can show the following assertions for some subsequence of (u_n):

$$\frac{\partial u_n^{(\ell)}}{\partial t} \to u_t^{(\ell)} \text{ weakly in } L^2(Q), \tag{4}$$

$$u_n \to u \text{ weakly in } X \text{ and strongly in } L^p\left(0,T,[W^{m-1,p}(\Omega)]^d\right), \tag{5}$$

$$< T(u_n), z^{(\ell)} > \to < T(u), z^{(\ell)} >, \forall z^{(\ell)} \in C_0^\infty(Q) \tag{6}$$

$$\liminf_n < T(u_n), u_n^{(\ell)} > \ge < T(u), u^{(\ell)} >, \forall n \in \mathbb{N} \tag{7}$$

and

$$-\infty < \varphi^{(\ell)}(u) \le \liminf_n \varphi^{(\ell)}(u_n) < \infty. \tag{8}$$

To show (4): Given $\varepsilon>0$, any $n \in \mathbb{N}$ and any $w_n = (w_n^{(1)}, \dots, w_n^{(d)}) \in Y_n$, put $w_n^{(\ell)} = \dfrac{u_n^{(\ell)} - v^{(\ell)}}{\varepsilon}$.

Since $v^{(\ell)}$ is arbitrary, $w_n^{(\ell)}$ is absolutely for a given $u_n^{(\ell)}$. Then (2) yields

$$\begin{aligned}&\left(\frac{\partial u_n^{(\ell)}}{\partial t}, w_n^{(\ell)}\right) + \left(T(u_n), w_n^{(\ell)}\right)\\ &-\frac{1}{\varepsilon}\left[\varphi^{(\ell)}\left(u_n(t)-\varepsilon w_n(t)\right)+\varphi^{(\ell)}\left(u_n(t)\right)\right]\\ &\le \left(f_n^{(\ell)}(t), w_n^{(\ell)}(t)\right).\end{aligned} \tag{9}$$

In particular, since $w_n^{(\ell)}$ is arbitrary we can write (9) in the form

$$\begin{aligned}&-\left(\frac{\partial u_n^{(\ell)}}{\partial t}, \frac{u_n^{(\ell)}(t-\varepsilon)-u_n^{(\ell)}(t)}{-\varepsilon}\right)\\ &-\left(T(u_n(t)), \frac{u_n^{(\ell)}(t-\varepsilon)-u_n^{(\ell)}(t)}{-\varepsilon}\right)\\ &\le \left|\frac{\varphi^{(\ell)}\left(u_n(t)\right)-\varepsilon\dfrac{\partial u_n(t)}{\partial t}-\varphi^{(\ell)}\left(u_n(t)\right)}{-\varepsilon}\right|\\ &-(f_n^{(\ell)}(t), \frac{u_n^{(\ell)}(t-\varepsilon)-u_n^{(\ell)}(t)}{-\varepsilon}).\end{aligned}$$

Allowing $\varepsilon \to 0$, *we get*

$$\begin{aligned}&-\left(\frac{\partial u_n^{(\ell)}(t)}{\partial t}, \frac{\partial u_n^{(\ell)}(t)}{\partial t}\right) - \left(T(u_n(t)), \frac{\partial u_n^{(\ell)}(t)}{\partial t}\right)\\ &\le \left|\varphi'^{(\ell)}(u_n(t))\right|\left\|\frac{\partial u_n^{(\ell)}(t)}{\partial t}\right\|_2 - \left(f_n^{(\ell)}(t), \frac{\partial u_n^{(\ell)}(t)}{\partial t}\right).\end{aligned} \tag{10}$$

where $\varphi'^{(\ell)}$ is the Gateaux derivative of $\varphi^{(\ell)}$ *at* $u_n(t)$.

Similarly, from (2), we get

$$\begin{aligned}&\left(\frac{\partial u_n^{(\ell)}(t+\varepsilon)}{\partial t}, \frac{u_n^{(\ell)}(t+\varepsilon)-u_n^{(\ell)}(t)}{\varepsilon}\right)\\ &+\left(T(u_n(t+\varepsilon)), \frac{u_n^{(\ell)}(t+\varepsilon)-u_n^{(\ell)}(t)}{\varepsilon}\right)\\ &\le \left|\frac{\varphi^{(\ell)}\left(u_n(t)-\varphi^{(\ell)}(u_n(t)(t+\varepsilon)\right)}{\varepsilon}\right|\\ &+\left(f_n^{(\ell)}(t+\varepsilon), \frac{u_n^{(\ell)}(t+\varepsilon)-u_n^{(\ell)}(t)}{\varepsilon}\right).\end{aligned}$$

Letting $\varepsilon \to 0$, *we have*

$$\begin{aligned}&\lim_{h\to 0}\left(\frac{\partial u_n^{(\ell)}(t+\varepsilon)}{\partial t}, \frac{\partial u_n^{(\ell)}(t+\varepsilon)}{\partial t}\right)\\ &+\lim_{h\to 0}\left(T(u_n(t+\varepsilon)), \frac{u_n^{(\ell)}(t+\varepsilon)-u_n^{(\ell)}(t)}{\varepsilon}\right)\\ &\le \left|\varphi'^{(\ell)}(u_n(t))\right|\left\|\frac{\partial u_n^{(\ell)}(t)}{\partial t}\right\|_2\\ &+\lim_{\varepsilon\to 0}\left(f_n^{(\ell)}(t+\varepsilon), \frac{\partial u_n^{(\ell)}(t)}{\partial t}\right).\end{aligned} \tag{11}$$

Adding (10), (11) and integrating over (0,t), taking into account A_2) we get

$$\begin{aligned}&\lim_{h\to 0}\frac{1}{2\varepsilon^2}\int_\Omega [u_n^{(\ell)}(\tau+\varepsilon)-u_n^{(\ell)}(\tau)]^2|_0^t dx\\ &\le \int_0^t \left\|f_n'^{(\ell)}(\tau)\right\|_2 \left\|\frac{\partial u_n^{(\ell)}(\tau)}{\partial t}\right\|_2 d\tau\\ &+2\int_0^t \left|\varphi'^{(\ell)}(u_n(\tau))\right|\left\|\frac{\partial u_n^{(\ell)}(\tau)}{\partial t}\right\|_2 d\tau.\end{aligned}$$

Therefore

$$\begin{aligned}&\left\|\frac{\partial u_n^{(\ell)}(t)}{\partial t}\right\|_2^2\\ &\le 2\int_0^t \left\|f_n'^{(\ell)}(\tau)\right\|_2 \left\|\frac{\partial u_n^{(\ell)}(\tau)}{\partial t}\right\|_2 d\tau\\ &+4\int_0^t \left|\varphi'^{(\ell)}(u_n(\tau))\right|\left\|\frac{\partial u_n^{(\ell)}(\tau)}{\partial t}\right\|_2 d\tau + \left\|\frac{\partial u_n^{(\ell)}(0)}{\partial t}\right\|_2^2.\end{aligned} \tag{12}$$

On the other hand, we get from(2)

$$\left(\frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(\mathbf{t})}{\partial \mathbf{t}}, \frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(\mathbf{t})}{\partial \mathbf{t}}\right) + \left(\mathbf{T}(\mathbf{u}_{\mathbf{n}}(\mathbf{t})), \frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(\mathbf{t})}{\partial \mathbf{t}}\right)$$
$$\leq \left|\varphi'^{(\ell)}(\mathbf{u}_{\mathbf{n}}(t))\right| \left\|\frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(\mathbf{t})}{\partial \mathbf{t}}\right\|_2 + ((\mathbf{f}_{\mathbf{n}}^{(\ell)}(\mathbf{t}), \frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(\mathbf{t})}{\partial \mathbf{t}}).$$

In particular,

$$\left\|\frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(0)}{\partial \mathbf{t}}\right\|_2^2 + \left(\mathbf{T}(\mathbf{u}_{\mathbf{n}}(0)), \frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(0)}{\partial \mathbf{t}}\right)$$
$$\leq \left|\varphi'^{(\ell)}(\mathbf{u}_{\mathbf{n}}(0))\right| \left\|\frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(0)}{\partial \mathbf{t}}\right\|_2 + \left(\mathbf{f}_{\mathbf{n}}^{(\ell)}(0), \frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(0)}{\partial \mathbf{t}}\right). \quad (13)$$

From (12) and (13),we may apply Gronwall's inequality to get the estimate

$$\left\|\frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(0)}{\partial \mathbf{t}}\right\|_2^2 \leq const. \left\|\mathbf{f}_{\mathbf{n}}^{(\ell)}(0)\right\|_2^2, \forall n \in \mathbb{N}. \quad (14)$$

Using A_1) and G), taking Young's inequality into account, we get

$$\left\|\frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}(\mathbf{t})}{\partial \mathbf{t}}\right\|_2^2 \leq const. \forall n \in \mathbb{N} \ and \ t \in [0,T]$$

and (4) follows. Assertion (5) is a direct consequence of A_2), G) and Aubin's lemma. Assertion (8) follows from the lower semicontinuity of $\boldsymbol{\varphi}^{(\boldsymbol{\ell})}$.

To prove (6) and (7), it suffices to show

$$\mathbf{limsup}_n \int_0^T (\mathbf{T}(u_n), \mathbf{u}_{\mathbf{n}}^{(\ell)} - \mathbf{u}^{(\ell)}) \mathbf{dt} \leq 0. \quad (15)$$

Since for any $v \in \boldsymbol{C}^{\mathbf{1}}(\mathbf{0}, \boldsymbol{T}, [\mathbf{C}_{\mathbf{0}}^{\infty}(\Omega)]^{\mathbf{d}})$ we may find a subsequence $(\mathbf{v}_{\mathbf{k}}) = (\mathbf{v}_{\mathbf{k}}^{(\mathbf{1})}, \ldots, \mathbf{v}_{\mathbf{k}}^{(\boldsymbol{d})}) \subset \cup_{\mathbf{n}=\mathbf{1}}^{\infty} \mathbf{Y}_{\mathbf{n}}$ such that $v_k \to u$ weakly in X, we get from(2)

$$\int_0^T \left(\frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}}{\partial \mathbf{t}}, \mathbf{u}_{\mathbf{n}}^{(\ell)}\right) \mathbf{dt} + \int_0^T \left(\mathbf{T}(u_n), \mathbf{u}_{\mathbf{n}}^{(\ell)} - \mathbf{u}^{(\ell)}\right) \mathbf{dt}$$
$$\leq \int_0^T \left(\frac{\partial \mathbf{u}_{\mathbf{n}}^{(\ell)}}{\partial \mathbf{t}}, \mathbf{v}_{\mathbf{k}}^{(\ell)}\right) \mathbf{dt} + \int_0^T \left(\mathbf{T}(u_n), \mathbf{v}_{\mathbf{k}}^{(\ell)} - \mathbf{u}^{(\ell)}\right) \mathbf{dt}$$
$$+\varphi^{(\ell)}(\mathbf{v}_{\mathbf{k}}) - \varphi^{(\ell)}(\mathbf{u}_{\mathbf{n}}) - \int_0^T \left(\mathbf{f}_{\mathbf{n}}^{(\ell)}, \mathbf{v}_{\mathbf{k}}^{(\ell)} - \mathbf{u}_{\mathbf{n}}^{(\ell)}\right) \mathbf{dt}.$$

Letting n→ ∞, keeping k fixed we have

$$< u_t^{(\ell)}, \mathbf{u}^{(\ell)} > + \mathbf{limsup}_n \int_0^T \left(\mathbf{T}(u_n), \mathbf{u}_{\mathbf{n}}^{(\ell)} - \mathbf{u}^{(\ell)}\right) \mathbf{dt}$$
$$\leq < u_t^{(\ell)}, \mathbf{v}_{\mathbf{k}}^{(\ell)} > + \mathbf{limsup}_n \int_0^T \left(\mathbf{T}(u_n), \mathbf{v}_{\mathbf{k}}^{(\ell)} - \mathbf{u}^{(\ell)}\right) \mathbf{dt}$$
$$+\varphi^{(\ell)}(\mathbf{v}_{\mathbf{k}}) - \mathbf{liminf}_n \, \varphi^{(\ell)}(\mathbf{u}_{\mathbf{n}}) - \int_0^T \left(\mathbf{f}_{\mathbf{n}}^{(\ell)}, \mathbf{v}_{\mathbf{k}}^{(\ell)} - \mathbf{u}^{(\ell)}\right) \mathbf{dt}.$$

Since the left hand side of this inequality is independent of k, allowing k→ ∞ we get (15) and (i) of the theorem follows.

To prove (ii) little arguments are needed. For this aim, define the truncated perturbation $\boldsymbol{g}_{\boldsymbol{k}}^{(\boldsymbol{\ell})}$ (x,t;u)by

$$g_k^{(\ell)}(x,t;u) = \begin{cases} k \dfrac{g^{(\ell)}(x,t;u)}{\left|g^{(\ell)}(x,t;u)\right|} & if \ \left|g^{(\ell)}(x,t;u)\right| > k \\ g^{(\ell)}(x,t;u) & otherwise. \end{cases}$$

From (i), there exists $u_k \in \mathbf{Y}$ **with** $\mathbf{u}_{\mathbf{k}}(\mathbf{t}) \in \mathbf{K}$ **a.e.**, $\mathbf{u}_{\mathbf{k}}(\mathbf{0}) = \mathbf{0}$ such that

$$\int_0^T \left(\frac{\partial \mathbf{u}_{\mathbf{k}}^{(\ell)}}{\partial \mathbf{t}}, \mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{k}}^{(\ell)}\right) \mathbf{dt} + \int_0^T \left(\mathbf{T}(u_k), \mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{k}}^{(\ell)}\right) \mathbf{dt}$$
$$+\varphi_k^{(\ell)}(v) - \varphi_k^{(\ell)}(\mathbf{u}_{\mathbf{k}}) \geq \int_0^T (f^{(\ell)}, \mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{k}}^{(\ell)}) \mathbf{dt},$$

for every $\mathbf{v}^{(\boldsymbol{\ell})} \in \boldsymbol{c}^{\mathbf{1}}(\mathbf{0}, \boldsymbol{T}, \mathbf{C}_{\mathbf{0}}^{\infty}(\Omega))$ for which

$$\varphi_k^{(\ell)}(v) < \infty$$

where

$$\varphi_k^{(\ell)}(u_k) = \int_Q G_0^{k(\ell)}(x,t;u_k(x,t)) dxdt$$

and

$$G_0^{k(\ell)}(x,t;r) = \int_0^r g_k^{(\ell)}(x,t;s) ds.$$

Using the subgradient inequality for $\boldsymbol{G}_{\mathbf{0}}^{\boldsymbol{k}(\boldsymbol{\ell})}(\mathbf{x}, \mathbf{t}; \mathbf{r})$ as a function of $\mathbf{r}$, we have

$$\int_0^T \left(\frac{\partial \mathbf{u}_{\mathbf{k}}^{(\ell)}}{\partial \mathbf{t}}, \mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{k}}^{(\ell)}\right) \mathbf{dt} + \int_0^T \left(\mathbf{T}(u_k), \mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{k}}^{(\ell)}\right) \mathbf{dt}$$
$$+\int_Q g_k^{(\ell)}(x,t;u_k)\left(\mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{k}}^{(\ell)}\right) dxdt$$
$$\geq \int_0^T (\mathbf{f}^{(\ell)}, \mathbf{v}^{(\ell)} - \mathbf{u}_{\mathbf{k}}^{(\ell)}) \mathbf{dt}.$$

The rest of the proof is more or less word for word as in (i).

Example: As an example which can be handled by our result, consider the variational inequalities associated with the following system

$$\frac{\partial u^{(\ell)}}{\partial t} - \sum_{|\alpha| \leq \mathbf{m}} (-1)^{|\alpha|} \mathbf{D}^{\alpha} \left(\left|\mathbf{D}^{\alpha} u^{(\ell)}\right|^{p-2}\right) \mathbf{D}^{\alpha} u^{(\ell)}(\mathbf{x}, \mathbf{t})$$
$$+u^{(\ell)} e^u = f^{(\ell)}(x,t), \qquad \ell = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{d}, \mathbf{p} \geq 2.$$

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