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On Geometrical Methods that Provide a Short Proof of Four Color Theorem

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Abstract In this article we introduce a short and comprehensive proof of four color theorem based on geometrical methods. At the end of the article we will provide a short proof of the De Bruijn Erdos theorem for locally finite infinite graphs.

Keywords: *four color theorem, geometrical methods, De Bruijn Erdos theorem*

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1. Introduction and Preliminaries

The Four Color Theorem is well known in the mathematical community. It states that any number of connected locally connected regions located in the plane (i.e' they are all subsets of R^2), intersecting only on their common boundaries, can be separated by four colors. In order to present a coherent mathematical view of the above theorem we need to state the following definitions and use the basic concepts and facts in topology. For more details on the basic concepts in topology we recommend the book, Topology (1970) by James Dugundji. Throughout this article the paths and closed curves are subsets of finite a graphs hence we use a type of digital geometry to prove some of the arguments. But many of the lemmas and theorems are stated and proved in the most general cases. The Jordan closed curve theorem as stated at [2] is the basis for many of the lemmas and theorems. When dealing with closed curves and paths we define the interior path of the given closed curve to be paths that except one or two points on the boundary the rest of the path lives in the interior of the closed curve. For a connected Graph G, its spectrum, spect(G) is the operator norm of the adjacency matrix. We say that two connected graphs G_1 and G_2 are equivalent, $G_1 \sim G_2$, if their corresponding adjacency matrix are unitary equivalent. this concept is used to define some useful definitions that will help proving the main theorem, theorem (2.21). the paths are all directed in same way. For example the paths located on closed curves have the same direction as the direction on the closed curve which is clockwise direction. Many of the sets we are going to deal with are connected, closed and bounded. These kinds of sets constitute a very important class of sets.

Definition 1.1 A compact connected subspace of a Hausdorf metric space is called continuum.

In particular here by region we mean locally connected continuum sub-set of R^2. To state equivalent version of four color theorem represent each of the regions by a vertex in R^2 and connect two points if and only if the corresponding regions intersect. This will provide us with a graph G. Now the four color theorem can be expressed equivalently to this situation as to associate a color to each of the vertices in such a way that no two vertices connected by an arc can have the same color. Note that without loss of generality we can assume that the above graphs are connected. In fact throughout this article all the graphs are connected. For a set $Q \subseteq R^2$, let Q^c be the complement of Q in R^2, ie, $Q^c = (x \in R^2$, x not in Q). Also we denote the interior of Q by int(Q) In final part of this article we are going to deal with locally finite infinite graphs. These are infinite graphs with the property that each vertex is connected to finitely many other vertices. For a set Q, we set Card(Q) to be equal to the cardinality of the set Q. So by our definition any two region A_2 and A_2 from the collection are either disjoint or $(A_1) \cap (A_2) = (\text{Bound}(A_1) \cap \text{Bound}(A_2) \neq \phi$ In the later case A_1 and A_2 have to take different colors. Finally by Jordan closed curve theorem S divides the space in two regions, where one of the regions $SC \supseteq S$ can be be mapped homeomorphically onto the closed disc corresponding to a circle. For simplifying the notation we write int(S) instead of writing int(SC).

The following definitions are well known

Definition 1.2 A subset of $S \subseteq R^2$ which is homeomorphic to a cicle is called closed curve. The above homeomorphism can be chosen so that can be extended to the homeomorphism taking $SC = S \cup int(S)$ onto the closed disc that corresponds to the above circle.

Similarly the the edge or the arc connecting two vertices of a graph is homeomorphic to a closed interval. A path is a sub continuum of R^2 which is the union of number of arcs. It is clear that the path is also homeomorphic to a closed interval. Suppose S is a closed curve which is the subset of a connected graph G. Let us adopt the following notations.

Definition 1.3 Given a connected graph G, we denote the set of its vertices by V (G)), the number of its vertices by n(G), and the set of its edges by E(G). Suppose S ⊆ G be a closed curve. We set SCG = SC∩G and SCintG, to be equal to the set SCG minus E(G) ie' SCintG = SCG∩(E(G))^c.

Note that the set SCintG is not necessarily a connected set. Let G be a connected graph and suppose $(a_i)_{i=1}^{i=n(G)} \subset G$ be the set of all vertices of G. Furthermore for the vertices located on a closed curve S ⊆ G the sequence $(a_i) \subseteq S$ of vertices have clockwise orientation, ie' a_{i+1} is located to the right of a_i. a_i and a_{i+1} are called neighbouring vertices if they are connected by an edge in V(G).

Definition 1.4 For a given connected graph G, The function f from V(G) to the set (1,2,3,4) is called colorization if for any two neighboring vertices a_i and a_{i+1}, $f(a_i) \neq f(a_{i+1})$.

Definition 1.5 Following the same notations as in the above we define Cl(G), to be the set of all colorizations for G.

For any two points x,y ∈ G, let d(x,y) be the usual Euclidean distance of x from y in R^2. Let us set the following notations, d(x,G) = min(d(x,y), y ∈ G) and d(G) =max(d(x,y), x, y∈G). For each x ∈ G let D(x,d(x,G)) be equal to the closed Disk with the center in x and radius d(x,G). Let us define $D(G) = \bigcap_{x \in G} \left(D\left(x, d\left(x, G\right)\right)\right)$. Then it is easy to check that D(G) is a closed set containing G.

Definition 1.6 Keeping the same notations as in the above, the vertex x ∈ G is called to belong to the exterior boundary of G, denoted by EBound(G), if x can be connected via a path in R^2 ∩ (G)^c to any point in D(G)^c =R^2 - D(G) = ((y ∈ R^2), y not in D(G)).

In the following arguments in order to shorten the notations for a given closed curve S we write *int(S)* instead of *int(SC)*.

Definition 1.7 Keeping the same notations as in the above, The closed curve $S \subseteq G$ is called maximal closed curve if and only if $S \subseteq$ *EBound(G)*. It is called max closed curve if given some other closed curve S_1, with int(S_1) ⊇ int(S), then S_1 = S.

Suppose S is a closed curve, $S \subseteq G$. Now let $x_1, x_2, ...x_m$ be number of vertices on $S \bigcap G$, where xi can be reached from x_{i-1} moving clockwise on encounter once we move from xi to x_{i+1} including the two end points. By $G(x_i, x_{i+1})$ we mean the set of all the vertices of $G \bigcap S$ that we encounter once we move from x_i to x_{i+1} excluding the two end points. Note that since we are moving clockwise on the closed curve we have the following identifications, $x_{m+1} = x_1, x_{m+2} = x_2, ...$ and so on. Note that moving clock wise along S, the intervals $\left[x_i, x_j\right]$, (respectively (x_i, y_i)) are well defined only if $|i-j| \leq m-1$. Suppose $a_1, a_2, ...a_m$ be the set of all the vertices on S moving clockwise on S. Then for each integer $1 \leq i$, the vertices a_i and a_{i+1} are called neighbouring vertices. Note that S is a metric space with the metric inherited from R^2 . Given two points $x_1, x_2 \subseteq S$ where x_2 is to the right side of x_1 (ie' moving clockwise from x_1 we reach x_2), we set an order $x_1 \leq x_2$. the closed and open intervals on the closed curve S have the same definition as the ones on real line, just with the points belong to S, ie, $[x, y] = (z \in S),\ x \leq z \leq y,$ and $(x, y) = (z \in S),\quad x < z < y.$ In order to identify the intervals on S clearly, we let the set $(a_i) \subseteq S$ to take their indices from the well known ring Z_m. The following theorem is going to be used throughout the rest of the article.

Theorem 1.8 Keeping the same notations as in the above, two max closed curves that are subsets of G can not intersect in more than one point,

Proof Suppose S_1 and S_2 are are max closed curves that are subset of G. Let us give clockwise orientation to the above closed curves. Since S_1 and S_2 are not equal there exits a point x in S_1 not belonging to S_2. Moving from point x along S_1 clockwise suppose S_1 intersect S_2 at vertices y_1 and y_2. With y_1 the first vertex on S_2 that S_1 will encounter and y_2 the last one. Now consider the paths, $[x, y_1] \subseteq S_1$ $[y_1, y_2] \subseteq S_2$, and $[y_2, x] \subseteq S_1$. It is clear that S_3 the union of these three, paths is a closed curve. In order to prove the theorem it is enough to show the following condition, $int(S_3) \supseteq (int(S_1) \bigcup int(S_2))$ holds. We call the above condition the U-condition. Let $v \in D(G)^c$ and w be a point in $int(S_1) \bigcup int(S_2)$. For U-condition to hold it is enough to show that any path s connecting v to w will intersect S_3. To complete the proof we need to prove the following lemmas.

Lemma 1.8.1 Keeping the above notations suppose $S_1 \supseteq [y_1, y_2] = s_1 = (S_1) \bigcap (S_2)$ then the U-condition holds. Futhermore S_3C contains s_1.

Proof Let $x \in int(S_1)$ and $v \in D(G)^c$. Suppose s is a path connecting v to x . Then s has to intersect the boundary of S_1. Hence s either hits $[y_2, y_1] \subseteq S_1$ or it hits $[y_1, y_2] \subseteq S_1$. Suppose in its way to v it hits $y \in [y_1, y_2] \subseteq S_1$ for the last time at point $y \in [y_1, y_2] \subseteq S_1$. Now using the fact that $int(S_1) \bigcap int(S_2) = \phi,$ at point y, s either enters the interior of S_1 or enters the interior of S_2. In the first case it will hit $[y_2, y_1] \subseteq S_1$ before reaching v. In the second case it will hit $[y_1, y_2] \subseteq S_2$ before reaching v. This implies that $int(S_1) \bigcup int(S_2) \subseteq int(S_3)$ thus the U-condition hold. Now we clam that $int(S_3) \supseteq (y_1, y_2) \subseteq S_1$. To show that suppose

$x \in (y_1, y_2) \subseteq S_1$ Then for $y \in int(S_1)$ there exists a path $s_2 = [x, y] \in int(S_1)$. But if x is in $(S_3C)^c$, then s_2 must intersect S_3. At this point the fact that S_3 does not intersects $(y_1, y_2) \subseteq S_1$ implies that $int(S_3) \supseteq (y_1, y_2) \subseteq S_1$.

Definition 1.8.2 The encounter of the closed curves S_i, i=1,2, as stated in the above lemma is called of type-I

Lemma 1.8.3 Keeping the above notations suppose that S_1 and S2 intersect only on two points y_1 and y_2. Furthermore assume that moving clockwise from x on S_1 we enter the interior of S2 at y_1 and leave it at y_2. Then the U-condition holds.

Proof Note, it is easy to see that moving clockwise on S2 it will enter the interior of S_1 at y_2 and leave it at y_1. Let us consider the following paths. $p_1 = ([y_1, y_2] \subseteq S_1)$ and $p_2 = ([y_2, y_1] \subseteq S_2)$. Next define the closed curve $S_4 = p_1 \cup p_2$. Considering S_1 and S4, note that $S_1 \cap S_4 = p_1$. This using lemma 1.8.1 implies that $int(S_3) \supseteq int(S_1)$. Similarly we can show that $int(S_3) \supseteq int(S_2)$, and this complete the proof.

Definition 1.8.4 The encounter of the closed curves Si, i= 1,2, as stated in the above lemma is called of type-II.

Lemma 1.8.5 Keeping the above notations suppose that S_1 and S2 intersect only on two points y_1 and y_2. Furthermore suppose $S_1 \cap int(S_2) = S_1 \cap int(S_2) = \phi$. Then the U-condition holds

Proof Let us define the closed curves S4 and S5, S4 being equal to the union of $[y_2, y_1] \subseteq S_2$ and $[y_1, y_2] \subseteq S_1$. S5 being equal to the union of $[y_1, y_2] \subseteq S_2$ and $[y_1, y_2] \subseteq S_1$. It is easy to see that $int(S_2) \cap int(S_4) = \phi$, hence by the above lemma $int(S_5) \supseteq int(S_4) \cup int(S_2) \cup ((y_2, y_1) \subseteq S_2)$ Where the directions of paths on Si, i= 1,2 are the clockwise directions on S_i. Since $S_1 \cap S_5 = s_1 = ([y_1, y_2] \subseteq S_1)$ by lemma 1.8.1, $int(S_3) \supseteq int(S_1) \cup int(S_2)$ and this complete the proof

Definition 1.8.6 The encounter of the closed curves Si, i=1,2, as stated in the above lemma is called of type-III

In general following the same procedure as in the above moving clockwise on S_1 from x, suppose S_1 intersect S2 first time at y_1 and last time at y_2. Then let us define the closed curve S3, to be equal to the union of $[y_2, y_1] \subseteq S_1$ and $[y_1, y_2] \subseteq S_2$ Now set $S_{2,1}$ to be equal to the union of $[y_1, y_2] \subseteq S_1$ and $[y_1, y_2] \subseteq S_2$. But $S_1 \cap S_{2,1} = [y_2, y_1] \subseteq S_1$. Thus since $int(S_1) \cap int(S_{2,1}) = \phi$, the encounter of S_1 and $S_{2,1}$ is of type-I. Hence by lemma 1.8.1 $int(S_3) \supseteq int(S_1)$, similarly we can show that $int(S_3) \supseteq int(S_2)$ and this complete the proof of theorem 1.8.

Lemma 1.9 Suppose S_1 is a closed curved which is the subset of a connected graph G. Then there exist a max closed curve $S \subseteq G$, such that $int(S)$ contains $int(S_1)$

Proof For a given closed curve $S_2 \subseteq G$ if $int(S_2) \supseteq (S_1)$ then we replace S_1 by S2. Next Using lemma 1.8 if S_1 intersect another closed curve $S_2 \subseteq G$ in more than two points, then by the above arguments there exits another closed curve $S_3 \subseteq G$ such that $int(S_3) \supseteq int(S_1) \cup int(S_2)$. Now proceed inductively, after finite number of steps we will end up having a closed curve $S \subseteq G$, such that if S intersects any other closed curve $S_k \subseteq G$ at more than one point then $int(S) \supseteq int(S_k)$. Further more if $S_l \subseteq G$ is any other closed curves with $int(S) \subseteq int(S_l)$, then $S = S_l$.

Theorem 1.10 Let G be a connected graph. Then the exist a set of max closed curves $(S_i)_{i=1}^{i=m}$, and paths $(s_i)_{i=1}^{i=m}$, such that G is equal to the union of the above paths and the set Q, $Q = \bigcup_{i=1}^{i=m} (S_iC \cap G)$. Furthermore EBound(G)=$\left(\bigcup_i (S_i)_{i=1}^{i=m}\right) \cup \left(\bigcup_i (s_j)_{j=1}^{j=m}\right)$

Proof In order to prove the above theorem we need the following definition and lemma

Let us consider a max closed curve $S_1 \subseteq G$, with G a connected graph. Suppose we have a sequence of max closed curves $S_i \subseteq G$, $i = 1, 2, ..., n.$ Suppose for each $i < n$, S_i is connected to S_{i+1}, via a path in G.

Definition 1.11 Keeping the same notation as in the above. The sequence of closed curves (S_i), i = 2,3, …, n are called descenders of S_1.

Lemma 1.12 Keeping the same notation as in the above, Let $S_i \subseteq G$, i =1.2.3,.. be a sequence of max closed curves. Then the above max curves can be connected to each other only by unique path s ⊆ G. Furthermore if S2 and S3 are both connected to S1 via paths in G, then S2 and S3 or any of their descenders can not be connected to each other via paths in G.

Proof Follows from the definiton of max closed curves and the arguments of lemma 1.8.

Note that the above statements holds if we consider the vertices of the paths connecting the max closed curves as max closed curves. In particular if we retract each one of the max closed curves to a point the resulting will be a tree.

Finally theorem 3.22 implies that any one of max closed curve or corresponding connecting paths between them are subset of *EBound(G)*. But by theorem(1.10) any vertex of $z \in G$ is either a member of $S_iC \cap G$ for some max closed curve $S_i \subseteq G$ or is a vertex on one of the paths $s_{i,j}$ connecting two max closed curves $S_i \subseteq G$ and $S_j \subseteq G$. This complete the proof of theorem 1.10.

The following lemmas can be verified immediately therefore we skip the proofs.

Lemma 1.13 Keeping the same notations as before. Given a path $s \subset G$, then the vertices on s can be separated by using two colors.

Lemma 1.14 Given a closed curve $S \subset G$, with G a connected graph. Then if the number of vertices on S, is even then the vertices on S can be separated by two colors. Otherwise three color would be sufficient to separate all the vertices on S.

At this point suppose That G is a connected graph and $S \subseteq G$ be a max closed curve. Now let $S_1 \subseteq G$ be another closed curve with $S_1C \subseteq SC$ and such that $S_1C \operatorname{int} G = \phi$. Then G is equal to the union of $\left(S_1C\right)^c$ and S1. Assuming that G is located on the surface of an sphere implies that there exists another connected graph G1 ~ G such that S1 is max closed curve in G1. We denote S1 a semi max close curve.

Definition 1.15 We say a connected graph G has property R. if there exist a colorization $cl \in Cl(G)$ such that on every semi max closed curve $S \subseteq G$, $Card\left(cl(S)\right) \leq 3$.

Now it is clear to see that if G1 is a connected graph with property R and G2 another connected graph such that G1 ~ G2, then G2 has property R too. Next let G be a connected graph and $S \subseteq G$ a closed curve. Let $\left(a_i\right)$ $i = 1, 2, ..., m$ be a sequence of all the vertices on S.

Definition 1.16 Let $a_i, a_j \in V(S)$. Given a path $s \subseteq SC \operatorname{int} G$ connecting a_i to b_j. Now consider the closed curve $S_1 \subseteq G$ that is the union of $\left(\left[a_i, a_j\right] \subseteq S\right)$ and *s*. We call the path s reducible if there exists another internal path $s_1 \subseteq SC \operatorname{int} G$, between two different points $c \leq d \in G[a,b] \subseteq S$, such that for the closed curve $S_2 = \left([c,d] \subseteq S\right) \cup s_1$, S_2C is a proper subset of S_1C. Otherwise s is called irreducible. suppose there exist two vertices $a_i, a_j \in V(S)$ and irreducible path in $int(S)$, connecting a_i to a_j. If $j - i > 1$ we say S is of type-I. Otherwise if j = i + 1, we say S is of type-II. If S is not of type-I or type-II, we say S is of type-III.

Lemma 1.17 Given a closed curve $S \subseteq G$ with G a connected graph. Let $\left(a_i\right)$, $i = 1, 2, ..., m$ be the set of vertices of S moving clockwise on S. Suppose there exists a vertex $a_j \subset S$, such that the connected component of SCG that contains a_j and does not contains any edges in S, contains another vertex $S \ni a_k \neq a_j$ then there exits an irreducible path in $SC \operatorname{int} G$ connecting a_j to a vertex $a_j \neq a_l \subseteq S$.

Proof If a connected component $G_1 \subseteq G$ as in the above contains a_j and another vertex $a_k \neq a_j$, then by the fact that G_1 is connected graph too, we have a path $s \subseteq G_1$ that connect a_j to a_k. Then it is clear that there exits a vertex $a_l \in S$ and irreducible path $s \subseteq G$, connecting a_j to a vertex $a_j \neq a_l \in S$.

At this point we are going to introduce an special subset $\Gamma \subset R^2$, $\Gamma = \left(n(G), Spect(G)\right)$, G a connected graph. We impose ordering ≤ on Γ by $\Gamma \ni (x_1, y_1) \leq (x, y_2) \in \Gamma$ if $x_1 \leq x_2$ and $y_1 \leq y_2$. It is easy to see that ≤ is partial ordering on Γ. Now using the fact that for every integer k the set *spect(k) = (spect(G)*, G a connected graph and n(G) = k) is a finite set, we can order the above set by the magnitude of its members. Hence set $spect(k) = \left(r_i\right)_{i=1}^{i=m_k}$. Where for i > j, $r_i > r_j$, r_1 = lower bound(spect(k)) r_{m_k} = upper *bound(spect(k)*. Also to facilitate our notations for a closed curve S ⊆ G we replace Cl(SCG) by Cl(S).

Lemma 1.18 Let S_1 and S2 be subsets of connected graph G both being max closed curves. Suppose the above two closed curves are connected by a path s ⊆ G or a common vertex q and that they have colorizations $cl_i \in Cl\left(S_i\right)$, i = 1,2. Let us de_ne connected graph $G_1 = S_1 \cup S_2 \cup s$. Then There exists a colorization $cl_i \in Cl\left(G_1\right)$ extending cl_1 or cl_2.

Proof By lemma 1.12 Points on s can be separated by two colors, Thus cl_1 can be extended to $S_1 \cup s$ in an obvious way. Suppose the point $b \in S_2 \cup s$ is connected by an edge in s to the point c 6= b in s. If $cl_1(b) = i = cl_2(b) = j$ then we are done. Otherwise if i ≠ j we define $cl_3 \in Cl(S_2)$ by $cl_3(a) = i$, if $cl_2(a) = j$, $cl_3(a) = j$, if $cl_2(a) = i$ or else $cl_3(a) = cl_2(a)$. Now cl_1 can be extended to G1 and we are done in this case. Finally If S_1 and S2 are connected by a common vertex q and $cl_1(q) = i = cl_2(q) = j$ then we are done. Otherwise repeating the above arguments we can define a colorization cl_3 on S2 ∪ S_1 = G_1 and we are done.

At this point note that every connected subset the connected graph G is also a connected graph. As before we define an interior graph to be a subgraph of G that does not contain any element from the set E(G).

Theorem 1.19 Given a connected graph G Suppose every max close curve, S ⊆ G has property R. Then G has a colorization.

Proof Suppose G is a connected graph and Let $\left(S_i\right)_{i=1}^{i=m}$ be the set of all max closed curves in G. Let $\left(s_j\right)_{j=1}^{j=l}$ be the set of all paths in G connecting max closed curves. Where some of these max closed curves consists of single vertex. Therefore by theorem 1.10 and lemma 1.12 we can assume that G is equal to the union of the max closed curves, the part of G they contain in their interior and the paths in G that connecting them. By lemma 1.12 there are unique pathes between the maximal closed curves and for any two maximal closed curve that are connected to a third one their descenders are not connected to each others via paths or common vertices. Now our assumption states that for any one of max closed curve S ⊆ G, SCG has property R. Thus each of the maximal closed curves $S_j \subseteq G$ has a colorization cl_j acting on S_jCG. Next let us begin from the max closed curve S_1 with a colorization $cl_1 \in S_1C \subseteq G$ and all max closed curves $S_{1,r} \subseteq G$ that are connected to S_1 by the paths $s_{1,r} \subseteq G$. For each of the max closed curves $S_{1,r}$, set $G_{1,r} = S_{i,r}CG \cup s_{1,r}$. Now by lemma 1.12, we can extend cl1 to become a colorization on $G_{2,r} = S_1CG \cup G_{1,r}$. Now repeating the above process from each of the max closed curves $S_{1,r}$, using lemma 1.18 we are going to extend cl_1 to become a colorization on each of the descenders of S_1 and ultimately after repeating the above process finitely many times we will get a colorization acting on G.

2. The Main Result

In this section we are going to state the main theorem and its proof.

Lemma 2.20 Suppose G is a connected graph and S a subset of G which is the max closed curve with SCG = G. Then S has property R.

Proof In order to prove the theorem we need to state and proof the following lemmas

Lemma 2.20.1 Keeping the same notations as in the above suppose S is of type-II. Then there are two vertices $a_i \in G(S)$, i = 1, 2 and an irreducible path $s_1 \subseteq int(S)$ connecting the above two vertices. Furthermore there exist two connected graphs G^1 and and G^1 each containing a max closed curve, $S_1 \subseteq G_1$, $S^1 \subseteq G^1$, such that $S^1CG^1 = G^1$ and $S^1CG^1 = G^1$. Finally we have $int(S) \supseteq int(int(S^1)\, int(S_1) \supseteq int(int(S^1)$ and $G \sim G_1$.

Proof Set $S^1 = s_1 \cup (|a_1, a_2|) \subseteq S$, then $S^1 \subseteq G$ is a closed curve. Next we can construct a path $s_1 \subseteq G^c$ connecting a_1 to a_2. Let $G^1 = S^1CG$, $G_1 = G^1 \cup s_2$ and $S_1 = s_1 \cup s_2$. It is clear that $S_1 \subseteq G_1$ is a max closed curve in G_1 and $S_1CG_1 = G_1$. Next following their constructions, $int(S_1) \supseteq int(S^1)$ and $int(S) \supseteq int(S^1)$.

Before proceeding to the next lemma we need to bring the following definition

Definition 2.20.2 Let G_1 and G_2 be a connected graph and $S_i \subseteq G_i$, i = 1,2 be a max closed curves with $S_iCG = G_i$, i=1,2. then we call Si a full closed curve. In particular if $G_1 \sim G_2$ then we write $S_1 \sim S_2$.

Lemma 2.20.3 Let $S \subseteq G$ be as in the above. Then S is either equivalent to type-I or to type-III closed curve.

Proof If S is type-I or type-III we are done. Otherwise following notations and arguments of lemma 2.20.1, consider the full closed curve $S_1 \subseteq G_1$ and $S^1 \subseteq G$. G_1 connected graph. We had $S \sim S_1$, $int(S_1) \supseteq int(S^1)$, $V(S_1) = V(s_1) = V(G) \cap s_1$, $int(S) \supseteq int(S^1)$. Furthermore any path in the interior of S_1 connecting to vertices of S_1 lives in the interior of S^1. Now if S_1 is of type-I or type-III we are done, otherwise proceeding as in the above we have two closed curve S_2 and $S^2 \subseteq G$ with the following properties. S_2 is a full max closed curve subset of a connected Graph $G_2 \sim G$, $int(S_2) \supseteq int(S^2)$, $int(S_1) \supseteq int(S^2)$, $int(S^1) \supseteq int(S^2)$ and the interior paths connecting vertices of S_2 live in the interior of S^2. Proceeding by induction and because G is a finite graph after finitely many steps we get the sequence of closed curves, S_i and S^i $1 \leq I \leq m_0$ with the following properties. $S_i \sim S$, $int(S_i) \supseteq int(S^i)$, $int(S^{i-1}) \supseteq int(S^i)$ and such that any interior paths connecting the vertices of S_i lives in the interior of $S^{i.}$ Hence by the fact that G is finite at some stage m_0, either G_{m_0} is of type-I or of type-III. The proof of the lemma is complete.

To complete the proof of the lemma we use induction on n(G). So suppose given an integer k such that the statement of lemma hold for all connected graph having only one max closed curve S with $n(G) \leq k$. Lets $a_1, a_2, \ldots a_m$ be a set of all vertices on $S \subseteq G$, where ai can be reached from a_{i-1} moving clockwise on S. If all connected component of G intersect S at most at one point, ie, S is of type-III. In this case by lemma 1.14 we can construct a colorization $cl \in Cl(S)$, $cl(S) \leq 3$. Now we want to extend cl to become a colorization on all G. Since for each of the connected components $G_i \subseteq G$, $n(G_i) \leq k$, then by induction and theorem 1.19, Gi has property R. Now using lemma 1.18 we can extend cl to become a colorization for G thus G has property R. Otherwise by the above lemma 2.20.3 we can assume that S is of type-I. Hence for some vertex aj 2 S there exits an irreducible interior path $s \subseteq SCintG$ from a_j to a_l, where a_j and al vertices of S, moving clockwise on S with j + 1 <l. Therefore we will get a closed curve $S_1 \subseteq G$ which is the union of s and the path $[a_l, a_j] \subseteq S$. Now note that S_1CG is a connected graph with $n(G_1 = S_1 C \cap G) = k$. Hence by induction assumption $G_1 = S_1CG$ has property R therefore there exists a colorization $cl \in Cl(G_1)$ such that $Card(cl(S_1)) \leq 3$. Now note that any connected component of G that contains a vertex in $G[a_j, a_l] \subseteq S$, does not intersect S at any other point. Hence using lemma 1.14 and lemma 1.18 we can extend cl to become a colorization on G with $Card(cl(S)) \leq 3$. This complete the proof of lemma 2.20.

Theorem 2.21 every connected graph G, has property R thus has a colorization.

Proof By lemma 2.20 each one of max closed curves has property R. Finally Theorem 1.19 completes the proof.

Finally we are going to bring a short proof of De Bruijn Erdos theorem for locally finite infinite planar graphs

Theorem(De Bruijn Erdos) Suppose G is a locally finite infinite graph. Then G has a colorization.

Proof Note that without loss of generality we can assume that G is connected. Let $v \in G$ be a vertex in G. Consider all the paths $s \in E(G)$ of length $m \in N$ from v to other vertices in G. The union of these paths is a finite connected subgraph $G_m \subseteq G$. By theorem(2.21), there exists a colorization $cl_m \in Cl(G_m)$. Next $Cl(G_m)$ can be considered as a subset of $R^{n(G_m)}$ with integer entries. Let us pick randomly one of these graphs say $G_1 = G_{m_0}$. Thus there exist an infinite subsequence $(m \leq mi)_i$ of (N), and a colorization $cl_{1,\infty} \in Cl(G^1)$ such that for each Gmi the restriction of clmi to G1 will agree with $cl_{1,\infty}$. Proceeding inductively we can construct an increasing sequence of finite subgraphs of G, $G^1 \subseteq G^2 \subseteq \ldots\ldots \subseteq G^k \subseteq \ldots$, where for each $\leq k$, Gk has a colorization $cl_{k,\infty}$, with the property that the restriction of $cl_{k+1,\infty}$ to G^k will be equal to $cl_{k,\infty}$. But $G = S(G^k)_k$, and this complete the proof.

3. Appendix

In this section we are going to proof some technical lemmas needed in the proof of number of the lemmas and theorems in the above

Theorem 3.22 Keeping the same notations as in the above. let $S \subseteq G$ be a max closed curve. Then $S \subseteq EBound(G)$.

Proof As we demostrated, if (S_i), i = 1,2, … m be the set of all max closed curves, then G is the union of the sets

S_iCG together with the paths $s_{i,j}$, i,j = 1, 2, …, m, where $s_{i,j}$ is the unique path connecting Si to S_j. By the finiteness of the Graph G we can assume that each of the max closed curves is a circle and the connecting paths are straight lines going through the centers of corresponding circles. Next for each of the circles Si there exists a circle $S_{i,l}$ with the same center but strictly larger than Si. By taking $S_{i,l}$ close enough to Si we can make sure that $S_{i,l}$ does not intersect any other max closed curves or the paths connecting them to any other max closed curve other than S_i. set $b_{i,j}$ be the intersection of $s_{i,j}$ with $S_{i,l}$. Now moving clockwise on $S_{i,l}$, consider two points $b_{i,j,1} < b_{i,j} < b_{i,j,2}$ on $S_{i,l}$. We can choose them close enough so that if we draw lines $s_{i,j,1}$ and $s_{i,j,2}$, parallel to $s_{i,j}$ intersecting $S_{j,l}$ at $b_{i,j,2}$ and $b_{i,j,1}$ respectively, with $b_{j,i,2} > b_{j,i,1}$, then $s_{i,j,k}$, k = 1, 2 intersect G at S_i and S_j only. Next let $a_{j,i,k}$, k = 1, 2, be the intersection of $s_{i,j,k}$, k = 1, 2 with S_j respectively. let us define the closed curve $S_{i,j,s}$ to be de_ned to be the union of $[a_{i,j,1}, a_{j,i,2}] \subseteq$ si,j,1, $[a_{j,i,2}, a_{j,i,1}] \subseteq S_j$, $[a_{j,i,1}, a_{i,j,2}] \subseteq s_{i,j,2}$ and $[a_{i,j,2}, a_{i,j,1}] \subseteq S_i$. Then it is clear that $S_{i,j,s}$ contains $s_{i,j}$ in its interior. At this point we assume that Si is connected to the sequence of max closed curves (S_j), j = 2, …, m_1, where each of the above closed curves is connected to Si only. Next for each integer $2 \le j \le m$ we define a loop $\Omega(i,j) \subseteq (G)^c$ to be a path which is the union of following paths, $\left[b_{i,j,1}, b_{j,i,2}\right] \subseteq s_{i,j,1}, \left[b_{j,i,2}, b_{j,i,1}\right] \subseteq S_j$ and $\left[b_{j,i,1}, b_{i,j,2}\right] \subseteq s_{i,j,2}$. Now suppose S_j is connected to more than one closed curve $S_k \subseteq G, \quad k = 2,..m_{1,j}$ Ordering the points of intersection of $s_{j,k}$ with S_j. Let $a_{j,k}$, $j = 1,2,..,m_1$ be the sequence of the above vertices. At this point we assume that The sequence of closed curves that are connected to S_j, j = 1,2, …, m_1 are not connected to any other max closed curve. Next for a fixed $2 \le j_1 \le m_1$ and $2 \le k_1 \le m_{1,j_1}$ we defined the loop $\Omega(j_1, k_1) \subseteq G^c$ to be a path going from $b_{j_1,k_1,1} \in S_{j_1,l}$ to $S_{k_1,l}$ and back to $S_{j_1,l}$. Now we want to extend the definition of loop to more complicated case. we want to define a path $\Omega(i,j_1) \subseteq G^c$, going from $b_{i,j_1,1} \in S_{i,l}$ to $S_{j_1,l}$ and back to $S_{i,l}$. Set $\Omega(i, j_1)$ to be equal to the union of the following paths, $\left[b_{i,j_1,1}, b_{j_1,i,2}\right] \subseteq s_{i,j_1,1}$, $\left[b_{j_1,i,2}, b_{j_1,i,1}\right] \subseteq S_{j_1}$, $\left[b_{j_1,i,1}, b_{i,j_1,2}\right] \subseteq s_{i,j_1,2}$ and $\bigcup_{k=2}^{k=m_{1,j_1}} \left(\Omega(j_1,k) \cup \left[b_{j_1,k,2}, b_{j_1,(k+1),1}\right]\right)$. Using our assumptions each one of the paths $\Omega(j_1,k)$., is well defined hence the above formula is going to identify the path $\Omega(i, j_1) \subseteq G^c$. We call the above formula the loop formula. We saw that the loop formula is valid for two special cases. We want to show that the loop formula will identify the path $\Omega(i, j_1) \subseteq G^c$ from $S_{i,l}$ to $S_{j_1,l}$ and back to $S_{i,l}$ in the most general case. By the loop formula we can identify $\Omega(1, j_1) \subseteq G^c$, if we can identify $\Omega(j_1, k)$ for all the max closed curves connected to S_{j_1} except Si. Similarly to identify $\Omega(j_1, k)$, it is enough to identify all the loops $\Omega(j_2, k)$ where $S_{j_2} \ne S_i$ is one of the max closed curves that are connected to S_{j_1}, furthermore for each $2 \le k$, S_k is one of the max closed curves that is connected to $S_{j_2} \ne S_{j_1}$. We call this the stage (j_1, j_2) of calculation. inductively we have stages (j_1, j_2, …, j_n) of calculation. If there exists some integer n0 such that for $n \ge n_0$ at every stage (j_1, j_2, …, j_n) we can calculate (j_n, k) for all max closed curves that are connected to S_{j_n} and that are different from $S_{j_{n-1}}$, then we are done. But because of the finiteness of G, there exists an integer n_0 such that for $n \ge n_0$, Sjn is only connected to $S_{j_{n-1}}$. This implies that at the stages (j_1, j_2, …, j_{n_0-1}) we can calculate (j_{n_0-1},k), so by induction we are done. This implies the following lemma

Lemma 3.22.1 Keeping the same notations as in the above. For every connected max closed curves Si and S_j there exists a path $\Omega(i, j) \subseteq G^c$ going from $S_{i,l}$ to $S_{j,l}$ and back to $S_{i,1}$.

To complete the proof of the theorem we have to show that for any point $x \in G$, and $v \in D(G)^c$, there exists a path $s \subseteq G^c$ connecting x to v. Next We need to employ some new notation. As we mentioned we say two max closed curves $S_1 \subseteq G$ and $S_2 \subseteq G$ are connected, if they are connected by the path $s_{1,2}$. By chain of max closed curves we mean a sequence (Si), i = 1, 2, …, m such that for each $i \le m$, Si and S_{i+1} are connected. Now without loss of generality we can assume $S_1 \ni x = a_{1,j_1,1}$ for some max closed curve $S_1 \subseteq G$, where S_{j_1}, is a closed curves connected to S_1. The other cases can be overcome using some minor technicalities. Now there exists a path $s_1 \subseteq (S_1)^c$ from a_{1,j_1} to $b_{1,j_1,1}$, furthermore by the structure of G we can assume that in fact $s_1 \subseteq G^c$. Also there is a path $s_2 \in S_{1,l}^c$ from $b_{1,j_1,1}$ to v. Moving from v to $b_{1,j_1,1}$ on s_2 suppose the first point that we intersect G is $y = a_{k,k+1} \in S_k$, Where S_k, $S_{k+1} \subseteq G$ are connected max closed curves. The other cases can be overcome with minor technicality. At this point without loss of generality we can assume that s_2 will intersects $S_{k,1}$ first time at $b_{k,l,1}$. But by the structure of G, the exists a chain of closed curves $\left(S_{i,1}\right)_{i=1}^{i=k}$ rom $S_{1,l}$ to $S_{k,l}$. Next let us define the path $s_3 \subseteq G^c$, by $s_3 = \bigcup_{i=j_1+1}^{i=2+j_1} \left(\Omega(1, i-1) \cup \left[b_{1,i-1,2}, b_{1,i,1}\right]\right)$. In this paths all the intervals are on $S_{1,l}$ and it takes point $b_{1,j_1,1}$ to point $b_{1,2,1}$. Note since we number the max closed curves connected to S_1 S1 by moving clockwise on S_1 by our definition, $S_2 = S_{2+j_1}$. Let us denote s_3 by $\aleph\left(b_{1,j_1,1}, b_{1,2,1}\right)$. (It is also clear that if none of the points in the interval $\left[b_{1,j_1,1}, b_{1,2,1}\right] \subseteq S_{1,l}$ were connected to another max closed curve then $\aleph\left(b_{1,j_1,1}, b_{1,2,1}\right) = \left[b_{1,j_1,1} b_{1,2,1}\right]$. Now for an integer $i \le k$, consider the path $s(i,i+2) \subseteq G^c$, that is theunion of

$\left[b_{1,j_1,1}, b_{1,2,1}\right] \subseteq S_{1,l}$ and $\aleph\left(b_{i+1,i,2}, b_{i+1,i+2,1}\right)$. Finally consider the paths $s_4 = \left(\bigcup_{i=2}^{i=k-2} s\left(i, i+2\right)\right) \subseteq G^c$, $s_5 = \left[b_{k,k+1,1}, v\right] \subseteq s \bigcap G^c$ and $s_6 = \left[a_{k,k+1}, b_{k,k+1}\right] \subseteq G^c$. At this end the path $s_7 = s_5 \bigcup s_6 \bigcup s_4 \bigcup s_3 \bigcup s_1$ is a path in G^c taking x to v and this complete the proof of the theorem.

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