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On the Analytic Curve of C^2 which is not Omitted by Every Fatou-Bieberbach Domain

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Abstract Let *C* be an irreducible (may be transendental) analytic curve whose genus is geater than 1. Then every Fatou-Bieberbach domain does not omit *C*.

Keywords: *fatou-bieberbach domain, hyperbolic cuve, transcendental algebraic type curve, kobayashi hyperbolic*

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1. Introduction

We call that a domain $\Omega \subsetneq \mathbf{C}^2$ is a Fatou-Bieberbach domain if Ω is biholomorphic to C^2.

We call that an irreducible algebraic curve *A* of C^2 is hyperbolic when $g(A) > 0$ or else $g(A) = 0$ and the set $A \cap L_\infty$ consists more than two different points, where $g(A)$ is the genus of *A* and L_∞ is the line at infinity.

In the previous paper [2], we proved the case *A* (theorem 2.1). In this paper, every Fatou-Bieberbach domain does not omit *C* which is the same in Abstract (Theorem 3.12). Moreover we prove that the property that an analytic curve of $\mathbf{C}^2$ does not omit every Fatou-Bieberbach domain is unchanged one by a transformation of $Aut(\mathbf{C}^2)$ (Theorem 3.7). Therefore every Fatou-Bieberbach domain does not omit *T*(*A*) with $T \in Aut(\mathbf{C}^2)$. And we prove also that there is an algebraic type curve of $\mathbf{C}^2$ which is isomorphic to an algebraic curve of any topological type such that it can not be transformed to an algebraic curve by any $T \in Aut(\mathbf{C}^2)$ (Proposition 3.3).

2. Preliminary

Theorem 2.1 (Theorem 2.1 in [2]). *Let A be a hyperbolic algebraic curve of* $\boldsymbol{C}^2$*. Then every Fatou-Bieberbach domain Ω does not omit A.*

Remark 2.2 (Theorem 2.2 in [2]). Let Ω_0 be every Fatou-Bieberbach domain in $\mathbf{C}^2$ arising from a polynomial automorphism, namely a polynomial basin. Then Ω_0 does not omit every algebraic curve of $\mathbf{C}^2$.

Theorem 2.3 (Theorem 4.1 in Buzzard and Fornaess [4]). *Let X be an arbitrary analytic curve of* $\boldsymbol{C}^2$*. Then there is a Fatou-Bieberbach domain Ω with* $\Omega \supset X$ *and* $\Omega \neq C^2$ *where some* $\Phi : \Omega \to C^2$ *is biholomorphic such that every nonconstant holomorphic map* $f : \boldsymbol{C} \to \boldsymbol{C}^2$ *intersects with Φ(X) at in_nite points and* $\boldsymbol{C}^2 - \Phi(X)$ *is Kobayashi hyperbolic.*

Example 2.4 (Example 9.7 in [6]). There is a Fatou-Bieberbach domain Ω0 such that $\overline{\Omega}_0$ namely the closure of Ω_0 omits an algebraic complex line *L*.

Proposition 2.5 (Proposition 3.1 in [2]). *For some transendental complex line* $\tilde{L}$ *such that there is a Fatou-Bieberbach domain Ω0 with* $\overline{\Omega}_0 \cap \tilde{L} = \emptyset$.

From Theorem 2.3, Example 2.4 and Proposition 2.5, following theorem is easy to see.

Theorem 2.6. *Let C be an analytic curve of* $\boldsymbol{C}^2$*. Then every Fatou-Bieberbach domain does not omit C or some (not all) Fatou-Bieberbach do-main* Ω_0 *omits C.*

3. Conclusion

Proposition 3.1. *Let* $\tilde{A}$ *be a transendental hyperbolic curve of C2 which is isomorphic to a hyperbolic algebraic curve A. If* $\tilde{A}$ *is transformed to an alge-braic hyperbolic curve A of* $\boldsymbol{C}^2$ *by* $T \in Aut(\boldsymbol{C}^2)$*, then every Fatou-Bieberbach domain does not omit* $\tilde{A}$.

Proof. We assume that there is a Fatou-Bieberbach domain Ω with $\Omega \cap \tilde{A} = \emptyset$. Then $T(\Omega) \cap A = \emptyset$. and *T*(Ω) is biholomorphic to $\mathbf{C}^2$ by $T \circ \Phi$, where Φ is a biholomorphic map of Ω to $\mathbf{C}^2$. Therefore *T*(Ω) is a Fatou-Bieberbach domain and it contradicts to Theorem 2.1.

Proposition 3.2. *Let X and Ω be the same of Theorem 2.3. Then Φ(X) can not be transformed to any algebraic curve A which is hyperbolic or non by any* $T \in Aut(\boldsymbol{C}^2)$*.*

Proof. We assume that Φ(*X*) is transformed to an algebraic curve *A* by some $T \in Aut(\mathbf{C}^2)$. As *T*(Ω) is a Fatou-bieberbach domain with $T \circ \Phi(X) = A$. Let $f : \mathbf{C} \to \mathbf{C}^2$ be a map to an algebraic complex line. Then $(T \circ \Phi(X)) \cap f(\mathbf{C})$ is a finite set of points at most and $\Phi(X) \cap (T{-}1 \circ f(\mathbf{C}))$ is also. It is a contradiction to Theorem 2.3.

Proposition 3.3 (cf. Proposition 3.12 in [2]). *There is a transcendental analytic curve of* $\mathbf{C}^2$ *which is isomorphic to an algebraic curve of any topo-logical type, that is an algebraic type curve of any type, such that it can not be transformed to an algebraic curve by any* $T \in Aut(\mathbf{C}^2)$*. If we take X an algebraic type curve of any topological type, Φ(X) which is the same notation of Theorem 2.3 is such one.*

Proposition 3.4. *Let X, Ω and Φ be the same of Theorem 2.3. Then every Fatou-Bieberbach domain Ω′ does not omit Φ(X).*

Proof. If some Ω′ omits Φ(*X*), $\Phi'^{-1} : \mathbf{C}^2 \to \Omega' \subset \mathbf{C}2-\Phi(X)$, where Φ′ is a biholomorphic map of Ω′ to $\mathbf{C}^2$. Since $\mathbf{C}^2-\Phi(X)$ is Kobayashi hyperbolic, Φ'^{-1} is a constant map. It is a contradiction.

Proposition 3.5. *Let C be an analytic curve of* $\mathbf{C}^2$ *which is transformed by* $T \in Aut(\mathbf{C}^2)$ *to some analytic curve C′ of* $\mathbf{C}^2$ *which does not omit every Fatou-Bieberbach domain. Then every Fatou-Bieberbach domain does not omit C.*

Proof. We assume that there is a Fatou-Bieberbach domain Ω_0 with $\Omega_0 \cap C = \emptyset$. Then $T(\Omega_0) \cap T(C) = \emptyset$. Since $T(\Omega_0)$ is a Fatou-Bieberbach domain which omits C′ = T(C). It contradicts to the property of C.

Corollary 3.6. *Let C be an analytic curve of* $\mathbf{C}^2$ *which does not omit every Fatou-Bieberbach domain. Then T(C) with every* $T \in Aut(\mathbf{C}^2)$ *does not omits every Fatou-Bieberbach domain.*

Proof. Since $T^{-1} \circ T(C) = C$, T(C) does not omit every Fatou-Bieberbach domain also by Proposition 3.5.

From Proposition 3.5 and Corollary 3.6 it is easy to see the following theorem.

Theorem 3.7. *The property that the analytic curve of* $\mathbf{C}^2$ *does not omit every Fatou-Bieberbach domain is unchanged one by a transformation of* $Aut(\mathbf{C}^2)$*.*

From Proposition 3.4 and Theorem 3.7 it is easy to see the following corollary.

Corollary 3.8. *Let C be an analytic curve of* $\mathbf{C}^2$ *which is transformed by some* $T \in Aut(\mathbf{C}^2)$ *to Φ(X) where X, Ω and Φ are the same of Proposition 3.4. Then every Fatou-Bieberbach domain Ω′ does not omit C.*

Problem 3.9. Is there $\tilde{A}$ which is not transformed to an algebraic hyperbolic curve *A* of $\mathbf{C}^2$ or Φ(*X*), which is the same notation of Theorem 2.3, by some $T \in Aut(\mathbf{C}^2)$?

Proposition 3.10. *Let C be an irreducible analytic curve of* $\mathbf{C}^2$ *with g(C) > 1. Then* $D = \mathbf{C}^2 - C$ *is not biholomorphic to* $\mathbf{C}^2$*, that is, D is not a Fatou-Bieberbach domain*.

Proof. We assume that there is a biholomorphic map Φ of *D* to $\mathbf{C}^2(x, y)$ such as $\Phi : x = \xi(z,w)$, $y = \eta(z,w)$ where $(z,w) \in D$. Let $P(x, y)$ be a nonconstant primitive polynomial, that is all level curve of {P(x, y) = α} is irreducible except at most finite number of $\alpha 1, \cdots, \alpha n$. Then $P(\xi(z,w), \eta(z,w)) \in \mathcal{O}(D)$. If *C* is an essential singular curve of $P(\xi, \eta)$, every level curve of $P(\xi, \eta)$ can not be analytically continued to every point of *C* except at most one value $\alpha 0$ by well known Thullen's theorem.

Since $\{P(\xi, \eta) = \alpha, \alpha \neq \alpha_0, \alpha_1, \cdots, \alpha_n\}$ is considered as a Riemann surface $R_0 = R - \{p_1, \cdots, p_m\}$ where R is a compact Riemann surface, that is an algebraic type Riemann surface such as $\pi : R_0 \to \{P(\xi, \eta) = \alpha\} \subset \mathbf{C}^2$ is the normalization. And at some $p_i \in \{p_1, \cdots, p_m\}$ the cluster set of π in $\mathbf{C}^2$ is C. Because it is a pseudoconcave set of $\mathbf{C}^2$ by Theorem 3.4 in [1] and it is a degeneration locus of Kobayashi pseudodistance by Theorem 3.6 in [1]. It is a contradiction to g(C) > 1 by Theorem 4 in [3].

Therefore *C* is not an essential singular curve of $P(\xi, \eta)$, that is, $P(\xi, \eta)$ is at most meromorphically continued to *C*. Since $P(x, y) \in \mathcal{O}(\mathbf{C}^2)$, $P(\xi, \eta)$ is holomorphically continued to *C*. We set such function as $F(z,w)$.

Since every level curve of $F(z,w)$ is holomorphically isomorphic to algebraic type Riemann surface and an analytic cuve of $\mathbf{C}^2$, $F(z,w)$ is an algebraic type entire function of Nishino's sence [5], namely in the class (*A*). Then by principal theorem of [5], $F = \varphi \circ Q \circ T$ where φ is a polynomial function of one complex variable because $P(x, y)$ is primitive, *Q* is a primitive polynomial and $T \in Aut(\mathbf{C}^2)$.

Then $T_{|D} = \Phi$. It contradicts to the assumption. Then *D* is not biholomorphic to $\mathbf{C}^2$.

Proposition 3.11. *Let C be an irreducible analytic curve of* $\mathbf{C}^2$ *such that* $D = \mathbf{C}^2 - C$ *is not a Fatou-Bieberbach domain. Then every subdomain of D′ of D is not a Fatou-Bieberbach domain.*

Proof. If *D′* is a Fatou-Bieberbach domain, there is a biholomorphic map Φ of $\mathbf{C}^2$ to D′. Let I be an inclusion map of D′ to D and {f = 0} = C where $f \in \mathcal{O}(\mathbf{C}^2)$. Since $g = f \circ I \circ \Phi$ is an entire function of $\mathbf{C}^2$ and $g \neq 0$. There is a complex line L of $\mathbf{C}^2$ and $g|_L$ is considered as a transendental entire function of **C**, it is a contradiction by little Picard theorem because $I \circ \Phi|_L$ is an one to one map.

From Proposition 3.10 and 11, following theorem is easy to see.

Theorem 3.12. *Let C be the same of Proposition 3.10. Then every Fatou-Bieberbach domain does not omit C.*

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