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Common Fixed Point Theorems for Two Self-maps in Cone Metric Spaces under Different Contractive Conditions

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Abstract The existence of unique common fixed point theorems for two weakly compatible self-maps satisfying different contractive conditions in cone metric spaces without using normality.

Keywords: *weakly compatible maps, common fixed points, cone metric spaces, coincidence points*

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1. Introduction

The fundamental work in metric fixed point theory, by Stefan Banach in 1922 is famous as Banach Contraction Principle and found many applications viz., in proving the existence and uniqueness of solutions of Differential, Integral, Integro-Differential, Impulsive differential equations, etc. A number of authors introduced different contractive type mappings and proved many fixed point theorems extending the theory. B.E.Rhoades [1], Paula Collaco and Jaime Carvalho E Silva [2] compared various definitions of contractive mappings. In 2007, Huang and Zhang [3] introduced the concept of cone metric spaces by replacing the codomain with Banach spaces in a metric function whose range satisfy the properties of a cone. Subsequently, Abbas and Jungck [4], Abbas and Rhoades [5] have studied common fixed point theorems in cone metric spaces for normal cones with the assumption of normality. Sh.Rezapour and R.Hamlbarani [6] proved some fixed point theorems for any cone, omitting the assumption of normality. Recently, several authors have proved and proving many common fixed point theorems. See [7-12]. The purpose of this paper is to prove the common fixed point theorems for all cones which were proved by Abbas and Jungck in [4] only for normal cones.

The following definitions are taken from [3,4,5]. In the entire paper, E *and* R represents Banach space and Real Numbers respectively.

Definition 1.1:([3]) Let E be a Banach space and $P \subseteq E$. P is called a cone if it satisfies the following properties

1.1(i) P is closed, non-empty and $P \neq \{0\}$.

1.1(ii) $a, b \in R, a, b \geq 0, x, y \in P$ *implies* $ax + by \in P$.

1.1(iii) $x \in P$ *and* $-x \in P$ *implies* $x = 0$.

Examples 1.2:

1.2(i) $E = R$, *and* $P = \{x \in E, x \geq 0\}$ is a cone.

1.2(ii) Every non-zero Banach Space acts as a cone to itself.

Definition 1.3: ([3]) For a given cone P, define a partial ordering $\leq$ w.r.t. P by

1.3(i) $x \leq y$ iff $y - x \in P$

1.3(ii) $x < y$ *implies* $x \leq y$ *and* $x \neq y$

1.3(iii) $x \ll y$ *implies* $y - x \in int\ P$, *where intP* denotes interior of P.

Definition 1.4:([3]) Let X be a non-empty set and E be a Banach Space. Suppose the mapping $d : XxX \to E$ satisfy

1.4(i) $0 \leq d(x, y)\ \forall\ x, y \in X$ *i.e.*, $d(x, y) \in P$

1.4(ii) $d(x, y) = 0$ *iff* $x = y$

1.4(iii) $d(x, y) = d(y, x)\ \forall\ x, y \in X$

1.4(iv) $d(x, y) \leq d(x, z) + d(z, y)\ \ \forall x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 1.5: $E = X = R, P = \{x \in E : x \geq 0\}$, and $d : XxX \to E$ *defined by* $d(x, y) = |x - y|$.

Then (X, d) is a cone metric space.

Definition 1.6: ([3]) Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in X. Then

1.6(i) $x_n \to x$ whenever for every $c \in E$ with $0 \ll c, \exists$ a natural number N such that $d(x_n, x) \ll c \forall n \geq N$.

1.6(ii) $\{x_n\}$ is a Cauchy Sequence if for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c \forall n, m \geq N$..

1.6(iii) (X, d) is a complete cone metric space if every Cauchy sequence in X is convergent in X.

The following lemma's are taken from [3].

Lemma 1.7: ([3]) Let (X,d) be a cone metric space. Let P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then

1.7(i) $x_n \to x$ iff $d(x_n,x) \to 0$ as $n \to \infty$.

1.7(ii) $\{x_n\}$ is a Cauchy sequence iff $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

Lemma 1.8: ([3]) Let (X,d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$ i.e., the limit of the sequence is unique.

For the definition and examples of normal and non-normal cones, see [3,4,5].

Definition 1.9: ([4]) Let X be a non-empty set and f,g two self maps on X. Then

1.9(i) If $q = fp = gp$ for some $p \in X$, then p is called a coincidence point of f and g. Also q is called a point of coincidence of f and g.

1.9(ii) If $p = fp = gp$ for some $p \in X$, then p is called a common fixed point of f and g.

Definition 1.10: ([4]) Let X be a non-empty set and f,g two self maps on X. The pair $\{f,g\}$ is said to be weakly compatible if $f(gt) = g(ft)$ whenever $ft = gt$ for some $t \in X$.

Lemma 1.11: ([4]) Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

2.Main Results

Theorem 2.1: Let (X,d) be a cone metric space. Suppose the self-maps $f,g : X \to X$ satisfy the contractive condition

$$d(fx,fy) \le kd(gx,gy) \quad \forall\, x,y \in X \qquad 2.1(1)$$

where $k \in [0,1)$ is a constant. If the range of f is contained in range of g and range of g is a complete subspace of X. More over if f and g are weakly compatible, f and g have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$. This possible since $f(X) \subseteq g(X)$. Continuing this process, choose $x_{n+1} \in X$ such that

$$f(x_n) = g(x_{n+1}) \qquad 2.1(2)$$

Now,

$$d(gx_{n+1},gx_n) = d(fx_n, fx_{n+1}) \le kd(gx_n,gx_{n-1})$$

using 2.1(1)

By Repeated application of 2.1(1), we get

$$d(gx_{n+1},gx_n) \le k^n d(gx_1,gx_0) \qquad 2.1(3)$$

For $n > m$,

$$\begin{aligned} d(gx_n,gx_m) &\le d(gx_n,gx_{n-1}) + d(gx_{n-1},gx_{n-2}) \\ &\quad + \ldots + d(gx_{m+1},gx_m) \\ &\le (k^{n-1}+k^{n-2}+\ldots+k^m)\, d(gx_1,gx_0) \qquad 2.1(4) \\ &\le k^n \left(1+k+k^2+\ldots+\ldots\right) d(gx_1,gx_0) \\ &= \frac{k^m}{1-k} d(gx_1,gx_0) \end{aligned}$$

Let $0 \ll c$ be given. Choose $n_1, \in N$ such that

$$\frac{k^m}{1-k}\; d(gx_1,gx_0) \ll c \forall m \ge n_1 \in N \qquad 2.1(5)$$

From 2.1(4) and 2.1(5), we get

$$d(gx_n,gx_m) \ll c \forall m,n \ge n_1 \in N$$

$\Rightarrow \{gx_n\}$ is a Cauchy sequence in $g(X)$, be def.of 1.6(ii)

$\Rightarrow \{gx_n\}$ is a convergent sequence in $g(X)$, since $g(X)$ is complete in X.

Let $\{gx_n\}$ is convergent to $q \in g(X)$. Consquently, there is

$$p \in X \text{ such that } gp = q \qquad 2.1(6)$$

For the same given $c \in E$, choose $n_2 \in N$ such that

$$d(gx_{n-1},gp) \ll \frac{c}{k} \;\forall\; n \ge n_2 \in N \qquad 2.1(7)$$

Hence, using 2.1(7)

$$d(gx_n, fp) = d(fx_{n-1}, fp) \le kd(gx_{n-1},gp) \ll c,$$

$$\Rightarrow d(gx_n, fp) \ll c \quad \forall \quad n \ge n_2 \in N$$

$$\Rightarrow gx_n \to fp.$$

Hence $\{gx_n\}$ converges to both q and fp. By uniqueness property of limit

$$fp = gp = q \qquad 2.1(8)$$

$\Rightarrow q$ is the point of coincidence of f and g.

Let $r \in X$ be any other coincidence point of f and g.

$\Rightarrow fr = gr$ be the point of coincidence of f and g 2.1(9)

Now, $d(gr,gp) = d(fr,fp) \le kd(gr,gp)$

$\Rightarrow d(gr,gp) \le kd(gr,gp)$

$\Rightarrow (k-1)d(gr,gp) \in P$ using 1.3(i)

Multiplying with positive real number $(1-k)$, we get $-d(gr,gp) \in P$. But, we have $d(gr,gp) \in P$.

From the definition of cone and cone metric, we get

$$gr = gp. \qquad 2.1(10)$$

From 2.1(8,9,10), f and g have unique point of coincidence.

Finally, let f and g are weakly compatible self-maps having unique point of coincidence. Using the lemma 1.11, f and g have a unique common fixed point.

Example 2.1.1: Let $X = E = R$, $P = \{x \in E : x \ge 0\}$,

$d: RxR \to E$ *such that* $d(x,y)=|x-y|$. Define

$$fx=\frac{\alpha}{\beta+1}x \text{ if } x\neq 0 \text{ and } fx=\gamma \text{ if } x=0,$$

$$gx=\alpha x \text{ if } x\neq 0 \text{ and } gx=\gamma \text{ if } x=0,$$

where α *is constant*, $\beta\geq 1$ *and* $\gamma\neq 0$.

Clearly $d(fx, fy)\leq kd(gx,gy)\ \forall\ x,y\in X$

where $k=\frac{1}{\beta}\in(0,1]$ is a cone metric space.

Moreover, $0\in X$ is coincidence point for f and g. But f and g are not weakly compatible at $0\in X$. Hence f and g do not have common fixed point.

Theorem 2.2 : Let (X,d) be a cone metric space. Suppose the maps $f,g: X\to X$ satisfy the contractive condition

$$d(fx, fy)\leq k\left[d(fx,gx)+d(fy,gy)\right]\ \forall x,y\in X \quad 2.2(1)$$

where $k\in[0,\frac{1}{2})$ is a constant. If the range of f is contained in range of g and the range of g is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof: Let $x_0\in X$ be arbitrary. Choose $x_1\in X$ such that $f(x_0)=g(x_1)$ whitch is possible since $f(X)\subseteq g(X)$. Continuing this process, choose $x_{n+1}\in X$ such that

$$f(x_n)=g(x_{n+1}) \quad 2.2(2)$$

Now,

$$\begin{aligned}
&d(g_{n+1}, gx_n = d\ \ fx_n\ \ fx_{n-1}\\
&\leq k[d(fx_n,gx_n)+d(fx_{n-1},gx_{n-1})]\\
&\leq k[d(gx_{n+1},gx_n)+d(gx_n,gx_{n-1})]\\
&\Rightarrow d(gx_{n+1},gx_n)\leq\frac{k}{1-k}d(gx_n,gx_{n-1})\\
&=hd(gx_n,gx_{n-1}),\ where\ h=\frac{k}{1-k}.
\end{aligned}$$

By repeated application of 2.2(1), we get

$$d(gx_{n+1},gx_n)\leq h^n\ d(gx_1,gx_0) \quad 2.2(3)$$

For $n>m$,

$$\begin{aligned}
&d(gx_n,gx_m)\leq d(gx_n,gx_{n-1})+d(gx_{n-1},gx_{n-2})\\
&\qquad +\ldots+d(gx_{m+1},gx_m)\\
&\leq\left(h^{n-1}+h^{n-2}+..+h^m\right)d(gx_1,gx_0) \quad 2.2(4)\\
&\leq h^m\left(1+h+h^2+...+\ldots\right)d(gx_1,gx_0)\\
&=\frac{h^m}{1-h}\ d(gx_1,gx_0),\ \text{since } |h|<1
\end{aligned}$$

Let $0\ll c$ be given. Choose $n_1\in N$ such that

$$\frac{h^m}{1-h}\ d(gx_1,gx_0)\ll c\ \forall\ m\geq n_1\in N \quad 2.2(5)$$

From 2.2(4), 2.2(5), we get

$$d(gx_n,gx_m)\ll c\forall m,n\geq n_1\in N$$

$\Rightarrow\{gx_n\}$ is a Cauchy sequence in $g(X)$.

$\Rightarrow\{gx_n\}$ is a convergent sequence in $g(X)$, since $g(X)$ is complete in X.

Let $\{gx_n\}$ is convergent to $q\in g(X)$. Consequently there is $p\in X$ such that

$$gp=q \quad 2.2(6)$$

For the same given $c\in E$, choose $n_2\in N$ such that

$$d(gx_n,gx_{n-1})\ll\frac{(1-k)c}{2k}$$

and

$$d(gx_n,gp)\ll\frac{(1-k)c}{2}\forall n\geq n_2\in N \quad 2.2(7)$$

Hence,

$$\begin{aligned}
&d(gx_n, fp)=d(fn_{n-1}, fp)\\
&\leq k[d(fx_{n-1},gx_{n-1})+d(fp,gp)]\\
&\leq k[d(gx_n,gx_{n-1})+d(fp,gx_n)+d(gx_n,gp)]\\
&\Rightarrow d(gx_n, fp)\leq\frac{1}{1-k}[kd(gx_n,gx_{n-1})+d(gx_n,gp)]
\end{aligned}$$

Applying 2.2(7), we get

$$\begin{aligned}
&d(gx_n, fp)\ll c\ \ \forall\ \ n\geq n_2\in N\\
&\Rightarrow gx_n\to fp
\end{aligned} \quad 2.2(8)$$

$\Rightarrow\{gx_n\}$ converges to both q and fp.

By uniqueness property of limit,

$$fp=gp=q. \quad 2.2(9)$$

$\Rightarrow\ fp=gp$ is point of coincidence of f and g.

Let $r\in X$ be any other coincidence point of f and g

$\Rightarrow\ fr=gr$ is the point of coincidence of f and g.

Now

$$\begin{aligned}
&d(gr,gp)=d(fr, fp)\\
&\leq k\left[d(fr,gr)+d(fp,gp)\right]=0\\
&\Rightarrow d(gr,gp)\leq 0\\
&\Rightarrow -d(gr,gp)\in P.
\end{aligned}$$

We have $d(gr,gp)\in P$.

From the definition of cone and cone metric, we get

$$gr=gp. \quad 2.2(10)$$

From 2.2(8,9,10), f and g have unique point of coincidence.

Finally, let f and g are weakly compatible self-maps having unique point of coincidence. Using the lemma 1.11, f and g have a unique common fixed point.

Example 2.2.1: Since $|x-y|\leq|x|+|y|\ \forall\ x,y\in R$, example 2.1.1 also satisfies the contractive condition 2.2(1) with the same $k=\frac{1}{\beta}\in(0,1]$.

Theorem 2.3: Let (X,d) be a cone metric space. Suppose the mappings $f,g:X\to X$ satisfy the contra

$$d(fx,fy)\le k\left[d(fx,gy)+d(fy,gx)\right]\ \forall\ x,y\in X \quad 2.3(1)$$

where $k\in[0,\frac{1}{2})$ s constant. If the range of f is contained in range of g and $g(X)$ is a complete subspace of X, then f and g have a unique coincidence point in X. Moreover, if f and g are weakly compatible, f and g have a unique common fixed point.

Proof: Let $x_0\in X$ be arbitrary. Choose $x_1\in X$ such that $f(x_0)=g(x_1)$ which is possible since $f(X)\subseteq g(X)$.

Continuing this process, choose $x_{n+1}\in X$ such that

$$f(x_n)=g(x_{n+1}) \quad 2.3(2)$$

Now,

$$\begin{aligned}
&d(gx_{n+1},gx_n)=d(fx_n,fx_{n-1})\\
&\le(fx_n,fx_{n-1})+d(fx_{n-1},gx_n)]\\
&\le k(d(gx_{n+1},gx_{n-1})+d(gx_n,gx_n)]\\
&\le k\left[d(gx_{n+1}),gx_n)+d(gx_n,gx_{n-1})\right]\\
&\Rightarrow d(gx_{n+1},gx_n)\le\frac{k}{1-k}\,d(gx_n\ \ gx_{n-1})\\
&=hd(gx_n,gx_{n-1}),\ where\ h=\frac{k}{1-k}.
\end{aligned}$$

By repeated application of 2.2(1), we get

$$d(gx_{n+1},gx_n)\le h^n\ d(gx_1,gx_0) \quad 2.3(3)$$

For $n>m$,

$$\begin{aligned}
&d(gx_n,gx_m)\le d(gx_n,gx_{n-1})+d(gx_{n-1},gx_{n-2})\\
&\qquad+\ldots+d(gx_{m+1},gx_m)\\
&\le\left(h^{n-1}+h^{n-2}+..+h^m\right)d(gx_1,gx_0) \quad 2.3(4)\\
&\le h^m\left(1+h+h^2+\ldots+\ldots\right)d(gx_1,gx_0)\\
&=\frac{h^m}{1-h}\ d(gx_1,gx_0),\ \text{since } |h|<1
\end{aligned}$$

Let $0\ll c$ be given. choose $n_1\in N$ such that

$$\frac{h^m}{1-h}d(gx_1,gx_0)\ll c\ \forall\ m\ge n_1\in N. \quad 2.3(5)$$

From 2.3(4), 2.3(5), we get

$$d(gx_n,gx_m)\ll c\ \forall\ m,n\ge n_1\in N$$

$\Rightarrow\{gx_n\}$ is a Cauchy sequence in $g(X)$

$\Rightarrow\{gx_n\}$ is a convergent sequence in $g(X)$, since $g(X)$ is complete in X.

Let $\{gx_n\}$ is convergent to $q\in g(X)$. Consequently there is $p\in X$ such that

$$gp=q \quad 2.3(6)$$

For the same given $c\in E$, choose $n_2\in N$ such that

$$d(gx_n,gx_{n-1})\ll\frac{c(1-k)}{2k}$$

and

$$d(gx_n,gp)\ll\frac{c(1-k)}{2k}\ \forall\ n\ge n_2\in N \quad 2.3(7)$$

Hence,

$$\begin{aligned}
&d(gx_n,fp)=d(fx_{n-1},fp)\\
&\le k\left[d(fx_{n-1},gp)+d(fp,gx_{n-1})\right]\\
&\le k\left[d(gx_n,gp)+d(fp,gx_{n-1})\right]\\
&\le k\left[d(gx_n,gp)+d(fp,gx_n)+d(gx_n,gx_{n-1})\right]\\
&\Rightarrow d(gx_n,fp)\le\frac{k}{1-k}[d(gx_n,gp)+d(gx_n,gx_{n-1})]\\
&\ll\frac{c}{2}+\frac{c}{2}=c\ \forall\ \ n\ge n_2\in N\ \text{from } 2.3(7)
\end{aligned}$$

$$\Rightarrow gx_n\to fp \quad 2.3(8)$$

$\Rightarrow\{gx_n\}$ converges to both q and fp.

By uniqueness property of limit,

$$fp=gp=q \quad 2.3(9)$$

$\Rightarrow$ $fp=gp$ is point of coincidence of f and g.

Let $r\in X$ be any other coincidence point of f and g

$\Rightarrow$ $fr=gr$ is the point of coincidence of f and g.

Now

$$\begin{aligned}
&d(gr,gp)=d(fr,fp)\le k\left[d(fr,gp)+d(fp,gr)\right]\\
&\le k[d(gr,gp)+d(gp,gr)]\\
&=2kd(gr,gp)\\
&\Rightarrow d(gr,gp)\le 2kd(gr,gp)\\
&\Rightarrow(2k-1)d(gr,gp)\in P
\end{aligned}$$

Multiplying with positive real number $(1-2k)$, we get $-d(gr,gp)\in P$. But we have $d(gr,gp)\in P$.

From the definition of cone and cone metric, we get

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Example 2.3.1: Let $X=E=R,\ P=\{x\in E:x\ge 0\}$, $d:RxR\to E$ *such that* $d(x,y)=|x-y|$.

Define $fx=\frac{x}{2}\ if\ x\ne 0\ and\ fx=\gamma\ if\ x=0$ and

$$gx=x\ if\ x\ne 0\ and\ gx=\gamma\ if\ x=0.$$

Clearly f and g satisfies the contractive condition 2.3(1) with $k=\frac{1}{3}\in(0,1]$. Moreover, $0\in X$ is coincidence point for f and g. But f and g are not weakly compatible at $0 \in X$. Hence f and g do not have common fixed point.

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References

[1] B.E.Rhoades, "A comparision of various definitions of contractive mappings", *Trans.Amer.Math.Soc.*26(1977) 257-290.

[2] Paula Collaco and Jaime Carvalho E Silva, "A complete comparision of 25 contraction conditions", *Nonlinear Analysis, Theory, Methods and Applications,* Vol.30, No.1, pp. 471-476.

[3] L.G.Huang, X.Zhang, "Cone metric spaces and fixed point theorems of contractive mappings", *J.Math.Anal.Appl.*332(2) (2007) 1468-1476.

[4] M.Abbas and G. Jungck, "Common fixed point results for non-commutating mappings withour continuity in cone metric spaces", *J.Math.Anal.Appl.* 341(2008)416-420.

[5] M.Abbas and B.E.Rhoades, "Fixed and Periodic point results in Cone metric spaces", *Appl.Math.Lett.*22(2009), 511-515.

[6] S.Rezapour and Halmbarani, "Some notes on the paper "cone metric spaces and fixed point theorems of contractive mappings",*J.Math.Anal.Appl.*345(2008)719-729.

[7] S.Rezapour, R.H.Hagi, "Fixed points of multi functions on cone metric spaces", *Number.Funct.Anal.* 30(7-9)(2009)825-832.

[8] Stojan Radenovic, "Common fixed points under contractive conditions in cone metric spaces", *Computers and Mathematics with Applications.* 58(2009)1273-1278.

[9] Vetro, "Common fixed points in cone meric spaces", *Rend.Ciric. Mat.Palermo.* 56(2007)464-468.

[10] T.Abdeljawad, E.Karapinar and K.Tas, "Common fixed point theorems in cone Banach spaces", *Hacet. J.Math.Stat.* 40(2011), 211-217.

[11] Jankovic, S.Dadelburg, Z.Radenovic, "On cone metric spaces, A survey", *Nonlinear Analysis,* 74(2011), 2591-2601.

[12] P.Raja, S.M.Vaezpour, "Some extensions of Banach's contraction principle in complete cone metric spaces", *Fixed Point Theory and Applications,* Article ID768294(2008), 11pages.