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Inequalities for the S^{th} Derivative of Polynomials Not Vanishing inside A Circle

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Abstract Let P(z) be a polynomial of degree n having all its zeros in $|z| \ge k, k \ge 1$, then for $1 \le R \le k$, Bidkham and Dewan [J. Math. Anal. Appl. 166(1992), 191-193] proved max

$$\max_{|z|=R}|P'(z)| \ge \frac{n(R+k)^{n-1}}{(1+k)^n}\max_{|z|=1}|P(z)|.$$

In this paper, we prove an interesting generalization as well as an improvement of this result by considering the s^{th} derivative of lacunary type of polynomials P(z) of degree n > 3.

Keywords: *derivative of a polynomial, zeros, exterior of circle, lacunary, inequalities*

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1. Introduction and Statement of Results

Let $P(z) := \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n and P'(z) its derivative, then it is known that

$$\max_{|z|=1}|P'(z)| \le \max_{|z|=1}|P(z)|. \qquad (1)$$

The above result, which is an immediate consequence of Bernstein's inequality on the derivative of a trigonometric polynomial is best possible with equality holding for the polynomial $P(z) = \lambda z^n$, where λ is a complex number.

If we restrict ourselves to the class of polynomials having all their zeros in $|z| \ge 1$, inequality (1) can be sharpened. In fact, Erdös conjectured and later Lax [6] proved that if P(z)≠ 0 in $|z| < 1$, then

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2}\max_{|z|=1}|P(z)|. \qquad (2)$$

On the other hand, if the polynomial P(z) of degree n has all its zeros in $|z| \le 1$, then it was proved by Turán [10], that max

$$\max_{|z|=1}|P'(z)| \ge \frac{n}{2}\max_{|z|=1}|P(z)|. \qquad (3)$$

Inequalities (2) and (3) are best possible and become equality for polynomials which have all zeros on $|z| = 1$.

Inequality (2) was refined by Aziz and Dawood [1] by showing that under the same hypothesis that

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2}\left\{\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\right\}. \qquad (4)$$

Equality in (4) holds for $P(z) = \alpha + \beta z^n, |\alpha| \ge |\beta|$.

For the class of polynomials P(z) of degree n having all their zeros in $|z| \ge k, k \ge 1$, Malik [7] proved:

$$\max_{|z|=1}|P'(z)| \le \frac{n}{1+k}\max_{|z|=1}|P(z)|. \qquad (5)$$

Inequality (5) was further improved by Govil [5] who under the same hypothesis proved:

$$\max_{|z|=1}|P'(z)| \le \frac{n}{1+k}\left\{\max_{|z|=1}|P(z)| - \min_{|z|=k}|P(z)|\right\}. \qquad (6)$$

Chan and Malik [3] obtained a generalization of (5) by considering the lacunary type of polynomials and obtained the following:

Theorem A: *Let* $P(z) := \sum_{j=\mu}^{n} a_j z^j$, $1 \le \mu \le n$ *be a polynomial of degree n having all its zeros in* $|z| \ge k, k \ge 1$, *then*

$$\max_{|z|=1}|P'(z)| \le \frac{n}{1+k^{\mu}}\max_{|z|=1}|P(z)|. \qquad (7)$$

The result is best possible and extremal polynomial is $P(z) = \left(z^{\mu} + k^{\mu}\right)^{\frac{n}{\mu}}$ *; where n is a multiple of* μ.

The next result was proved by Pukhta [8], who infact proved:

Theorem B: *Let* $P(z):=\sum_{j=\mu}^{n}a_j z^j$, $1\le\mu\le n$ *be a polynomial of degree n having all its zeros in* $|z|\ge k, k\ge 1$, *then*

$$\max_{|z|=1}|P'(z)|\le\frac{n}{1+k^{\mu}}\left\{\max_{|z|=1}|P(z)|-\min_{|z|=k}|P(z)|\right\}. \quad (8)$$

The result is best possible and extremal polynomial is $P(z)=\left(z^{\mu}+k^{\mu}\right)^{\frac{n}{\mu}}$ *; where n is a multiple of* μ .

Bidkham and Dewan [2] obtained a generalization of (5) by proving the following result:

Theorem C: *Let P(z) be a polynomial of degree n having all its zeros in* $|z|\ge k, k\ge 1$, *then for* $1\le R\le k$

$$\max_{|z|=1}|P'(z)|\ge\frac{n(R+k)^{n-1}}{(1+k)^n}\max_{|z|=1}|P(z)|. \quad (9)$$

The result is best possible and equality holds for P(z) = $(z + k)^n$.

In this paper, we prove the following generalization as well as an improvement of Theorem C by considering the s^{th} derivative of P(z).

Theorem 1: *If* $P(z):=\sum_{j=\mu}^{n}a_j z^j$, *n>3,* $1\le\mu\le n$ *is a polynomial of degree n having all its zeros in* $|z|\ge k, k\ge 1$, *then for* $1\le R\le k$, *and* $1\le s\le n$

$$\max_{|z|=1}\left|P^s(z)\right|\le \frac{n(n-1)...(n-s+1)}{R^s+k^s}\left\{\frac{\left(R^{n-1}-1\right)\left(n|a_0|+\mu|a_\mu|k^{\mu+1}\right)}{+n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)}-\frac{2}{(n+1)}|P'(0)|\left(\frac{\left(R^n-1\right)}{n}-(R-1)\right)\right\}$$
$$-|P''(0)|\left[\left(\frac{\left(R^n-1\right)-n(R-1)}{n(n-1)}\right)-\left(\frac{\left(R^{n-2}-1\right)-(n-2)(R-1)}{(n-2)(n-3)}\right)\right]$$
$$-\frac{n(n-1)...(n-s+1)}{R^s+k^s}\min_{|z|=k}|P(z)|.$$

2. Lemmas

For the proof of above theorem, we need the following lemmas. The first result is due to Qazi [9, Lemma 1].

Lemma 1: *If* $P(z):=a_0+\sum_{j=\mu}^{n}a_j z^j$, $1\le\mu\le n$ *is a polynomial of degree n having all its zeros in* $|z|\ge k, k\ge 1$, *then*

$$\max_{|z|=1}\left|P^s(z)\right|\le n\left\{\frac{n|a_0|+\mu|a_\mu|k^{\mu+1}}{n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\max_{|z|=1}|P(z)|.$$

The next lemma is due to Dewan, Kour and Mir [4].

Lemma 2: *Let* $P(z):=\sum_{j=0}^{n}a_j z^j$ *be a polynomial of degree n, then for* $R\ge 1$*;*

$$\max_{|z|=1}|P(z)|\le R^n\max_{|z|=1}|P(z)|-2\frac{\left(R^n-1\right)}{n+2}|P(0)| -\left[\frac{\left(R^n-1\right)}{n}-\frac{\left(R^{n-2}-1\right)}{n-2}\right]|P'(0)|, n>2 \quad (10)$$

and

$$\max_{|z|=1}|P(z)|\le R^n\max_{|z|=1}|P(z)|-\frac{(R-1)}{2}\left[\begin{array}{l}(R+1)|P(0)|\\+(R-1)|P'(0)|\end{array}\right], n=2. \quad (11)$$

Lemma 3: *If* $P(z):=a_0+\sum_{j=\mu}^{n}a_j z^j$, *n > 3,* $1\le\mu\le n$ *is a polynomial of degree n having all its zeros in* $|z|\ge k, k\ge 1$, *then for* $R\ge 1, 0\le\theta\le 2\pi$, *we have*

$$\left|P'\left(\mathrm{Re}^{i\theta}\right)\right|\le n\left\{\frac{nR^{n-1}\left(n|a_0|+\mu|a_\mu|k^{\mu+1}\right)}{n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)}\right\}\max_{|z|=1}|P(z)| -2\frac{\left(R^{n-1}-1\right)}{n+1}|P'(0)|-\left[\frac{\left(R^{n-1}-1\right)}{n-1}-\frac{\left(R^{n-3}-1\right)}{n-3}\right]|P''(0)|$$

Proof of Lemma 3: Since P(z) is a polynomial of degree n > 3, the polynomial P′(z) is of degree n ≥ 3, hence on applying inequality (10) of Lemma 2 to the polynomial P′(z), we obtain

$$\left|P'\left(\mathrm{Re}^{i\theta}\right)\right|\le R^{n-1}\max_{|z|=1}|P(z)| -2\frac{\left(R^{n-1}-1\right)}{n+1}|P'(0)|-\left[\frac{\left(R^{n-1}-1\right)}{n-1}-\frac{\left(R^{n-3}-1\right)}{n-3}\right]|P''(0)|$$

This proves Lemma 3.

Lemma 4: *If* $P(z):=a_0+\sum_{j=\mu}^{n}a_j z^j$, *n > 3,* $1\le\mu\le n$ *is a polynomial of degree n having all its zeros in* $|z|\ge k, k\ge 1$, *then for R ≥ 1,*

$$\max_{|z|=R}|P(z)| \le \left\{ \frac{\begin{array}{l}\left(R^{n-1}-1\right)\left(n|a_0|+\mu|a_\mu|k^{\mu+1}\right)\\ +n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)\end{array}}{n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)} \right\} \max_{|z|=1}|P(z)|$$
$$-\frac{2}{(n+1)}|P'(0)|\left\{\frac{R^n-1}{n}-(R-1)\right\}$$
$$-|P''(0)|\left[\left(\frac{\left(R^n-1\right)-n(R-1)}{n(n-1)}\right)-\left(\frac{\left(R^{n-2}-1\right)-(n-2)(R-1)}{(n-2)(n-3)}\right)\right]$$

Proof of Lemma 4: For each $0 \le \theta \le 2\pi$ and for $R \ge 1$, we have

$$P\left(Re^{i\theta}\right)-P\left(e^{i\theta}\right)=\int_1^R e^{i\theta} P'\left(r e^{i\theta}\right)dr.$$

Hence

$$\left|P\left(Re^{i\theta}\right)-P\left(e^{i\theta}\right)\right| = \int_1^R P'\left(r e^{i\theta}\right)dr, \qquad (12)$$

which when combined with Lemma 3, gives

$$\left|P\left(Re^{i\theta}\right)-P\left(e^{i\theta}\right)\right| = \int_1^R \left\{ \begin{array}{l} \dfrac{nr^{n-1}\left(n|a_0|+\mu|a_\mu|k^{\mu+1}\right)}{n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)}\max_{|z|=1}|P(z)| \\ -2\dfrac{\left(r^{n-1}-1\right)}{n+1}|P'(0)| \\ -\left[\dfrac{\left(r^{n-1}-1\right)}{n-1}-\dfrac{\left(r^{n-3}-1\right)}{n-3}\right]|P''(0)| \end{array} \right\} dr,$$

which gives

$$\max_{|z|=R}|P(z)| \le \frac{\begin{array}{l}\left(R^{n-1}-1\right)\left(n|a_0|+\mu|a_\mu|k^{\mu+1}\right)\\ +n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)\end{array}}{n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)} \max_{|z|=1}|P(z)|$$
$$-\frac{2}{(n+1)}|P'(0)|\left\{\frac{R^n-1}{n}-(R-1)\right\}$$
$$-|P''(0)|\left[\left(\frac{\left(R^n-1\right)-n(R-1)}{n(n-1)}\right)-\left(\frac{\left(R^{n-2}-1\right)-(n-2)(R-1)}{(n-2)(n-3)}\right)\right]$$

Hence the proof.

Lemma 5: *If P(z) is a polynomial of degree n having all its zeros in* $|z| \ge k, k \ge 1$*, then for* $1 \le s \le n$.

$$\max_{|z|=1}\left|P^s(z)\right| \le \frac{n(n-1)...n(n-s+1)}{1+k^s}\left\{\max_{|z|=1}|P(z)|-\max_{|z|=k}|P(z)|\right\}.$$

This Lemma is due to Govil [5].

Proof of Theorem 1: Since P(z) has all its zeros in $|z| \ge k, k \ge 1$ and if $1 \le R \le k$, then G(z) = P(Rz) has all its zeros in $|z| \ge \frac{k}{R}, \frac{k}{R} \ge 1$, therefore by applying Lemma 5 to G(z), we obtain

$$\max_{|z|=1}\left|G^s(z)\right| \le \frac{n(n-1)...n(n-s+1)}{1+\left(\frac{k}{R}\right)^s}\left\{\max_{|z|=1}|G(z)|-\max_{|z|=\frac{k}{R}}|G(z)|\right\}.$$

which implies

$$R^s\max_{|z|=1}\left|G^s(Rz)\right| \le \frac{n(n-1)...n(n-s+1)}{1+\left(\frac{k}{R}\right)^s}\left\{\max_{|z|=1}|G(Rz)|-\max_{|z|=\frac{k}{R}}|G(Rz)|\right\}.$$

which is equivalent to

$$\max_{|z|=R}\left|G^s(z)\right| \le \frac{n(n-1)...n(n-s+1)}{R+k^s}\left\{\max_{|z|=R}|G(z)|-\max_{|z|=k}|G(z)|\right\}. \qquad (13)$$

Inequality (13) in conjunction with Lemma 4 yields

$$\max_{|z|=R}\left|P^s(z)\right| \le \frac{n(n-1)...(n-s+1)}{R^s+k^s}\left\{ \frac{\begin{array}{l}\left(R^{n-1}-1\right)\left(n|a_0|+\mu|a_\mu|k^{\mu+1}\right)\\ +n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)\end{array}}{n|a_0|\left(1+k^{\mu+1}\right)+\mu|a_\mu|\left(k^{\mu+1}+k^{2\mu}\right)} \right\} \max_{|z|=1}|P(z)|$$
$$-\frac{2}{(n+1)}|P'(0)|\left\{\frac{R^n-1}{n}-(R-1)\right\}$$
$$-|P''(0)|\left[\left(\frac{\left(R^n-1\right)-n(R-1)}{n(n-1)}\right)-\left(\frac{\left(R^{n-2}-1\right)-(n-2)(R-1)}{(n-2)(n-3)}\right)\right]$$
$$-\frac{n(n-1)...(n-s+1)}{R^s+k^s}\min_{|z|=k}|P(z)|.$$

The proof of Theorem 1 is completed.

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