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He-Laplace Method for the Solution of Two-point Boundary Value Problems

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Abstract The boundary value problems of ordinary differential equations play an important role in many fields. Here, we implement the He-Laplace method for the solution of linear and nonlinear two-point boundary value problems. The aim of this paper is to compare the performance of the He-Laplace method with shooting method. As a result, for the same number of terms, our method provides relatively more accurate results with rapid convergence than other methods.

Keywords: *two-point boundary value problems, laplace transform, homotopy perturbation method, shooting method*

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1. Introduction

There are many new analytical approximate methods to solve two-point boundary value and initial value problems in the literature. These problems generally arise frequently in many areas of science and engineering, for example, fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aero-dynamics, reaction-diffusion process, geophysics etc. Different numerical methods have been proposed by various researchers. Among these are the Adomian decomposition method (ADM) [1,2,3,23,35,36,37], Homotopy perturbation method HPM [7-17,19,20,32,33,39], He-Laplace method [21,22,23,26,27,28,29,30]. There are many techniques available for the numerical solution of two-point boundary value problems for ordinary differential equations [18,25] and references therein. However there is no unified method to handle all types of nonlinear problems. Mohsen and El-Gamel [31] studied the performance of the collocation and Galerkin methods using Sinc basis functions for solving linear and nonlinear second-order two-point boundary value problems. The Sinc-collocation method is used to solve a system of second-order boundary value problems [5].

In this paper, we implement the He-Laplace method for solving the two-point boundary value problem of the form:

$$y'' = f(x, y, y'),\ \ a < x < b \tag{1}$$

with boundary conditions

$$y(a) = \alpha\ \ ,\ \ y(b) = \beta \tag{2}$$

where f is continuous on the set $D = \{\ (x, y, y') \,|\, a \le x \le b,\ y, y' \in R\}$, a, b, α, β are constants. The nonlinear shooting method [22] and He-Laplace method are used to find the solutions of the nonlinear two-point boundary value problems (1)-(2). In this paper, we present some numerical examples including linear as well as nonlinear boundary value problems. The main purpose of this paper is to compare the solutions obtained by He-Laplace method with Extended Adomian decomposition method (EADM) and Shooting method.

This paper contains basic idea of homotopy perturbation method and He-Laplace method in section 2, Numerical examples in section 3 and conclusion in section 4 respectively.

2. Basic Idea of Homotopy Perturbation Method

Consider the following nonlinear differential equation

$$A(y) - f(r) = 0\ ,\ \ r \in \Omega \tag{3}$$

with the boundary conditions of

$$B\left(y, \frac{\partial y}{\partial n}\right) = 0,\quad r \in \Gamma, \tag{4}$$

where A, B, $f(r)$ and Γ are a general differential operator, a boundary operator, a known analytic function and the boundary of the domain Ω, respectively.

The operator A can generally be divided into a linear part L and a nonlinear part N. Eq. (3) may therefore be written as:

$$L(y) + N(y) - f(r) = 0, \tag{5}$$

By the homotopy technique, we construct a homotopy $v(r, p) : \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p)=(1-p)\left[L(v)-L(y_0)\right]+p\left[A(v)-f(r)\right]=0$$

or

$$H(v,p)=L(v)-L(y_0)+pL(y_0)+p\left[N(v)-f(r)\right]=0$$

where $p\in[0,1]$ is an embedding parameter, while y_0 is an initial approximation of Eq.(1), which satisfies the boundary conditions. Obviously, from above equations, we will have:

$$H(v,0)=L(v)-L(y_0)=0, \quad (6)$$

$$H(v,1)=A(v)-f(r)=0, \quad (7)$$

The changing process of p from zero to unity is just that of $v(r,p)$ from y_0 to $y(r)$. In topology, this is called deformation, while $L(v)-L(y_0)$ and $A(v)-f(r)$ are called homotopy. If the embedding parameter p is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of Eqs.(6) and (7) can be written as a power series in p:

$$v=v_0+pv_1+p^2v_2+p^3v_3+\dots\dots\infty \quad (8)$$

Setting $p=1$ in Eq.(8), we have

$$y=\lim_{p\to 1} v=v_0+v_1+v_2+\dots\dots \quad (9)$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The series (9) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He [35] made the following suggestions:

(1) The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e. $p\to 1$.

(2) The norm of $L^{-1}\left(\frac{\partial N}{\partial v}\right)$ must be smaller than one so that the series converges.

3. He-Laplace Method

Consider the following nonlinear differential equation (IVP):

$$y''+p_1y'+p_2y+p_3f(y)=f(x) \quad (10)$$

$$y(0)=\alpha,\quad y'(0)=\beta \quad (11)$$

where $p_1, p_2, p_3, \alpha, \beta$ are constants. f(y) is a nonlinear function and f(x) is the source term. Taking Laplace transformation (denoted throughout this paper by L) on both side of Eq.(10), we have

$$L[y'']+L[p_1y']+L[p_2y]+L[p_3f(y)]=L[f(x)] \quad (12)$$

By using linearity of Laplace transformation, the result is

$$L[y'']+p_1L[y']+p_2L[y]+p_3L[f(y)]=L[f(x)] \quad (13)$$

Applying the formula on Laplace transform, we obtain

$$\begin{aligned}&s^2L[y]-sy(0)-y'(0)+p_1\{sL[y]-y(0)\}\\&+p_2L[y]+p_3L[f(y)]=L[f(x)]\end{aligned} \quad (14)$$

Using initial conditions in Eq.(14), we have

$$\begin{aligned}&(s^2+p_1s)L[y]\\&=\alpha s+\beta+\alpha p_1-p_2L[y]-p_3L[f(y)]+L[f(x)]\end{aligned} \quad (15)$$

Or

$$\begin{aligned}L[y]&=\frac{(\alpha s+\beta+\alpha p_1)}{(s^2+p_1s)}-\frac{p_2}{(s^2+p_1s)}L[y]\\&-\frac{p_3}{(s^2+p_1s)}L[f(y)]+L[f(x)]\end{aligned} \quad (16)$$

Taking inverse Laplace transform, we have

$$\begin{aligned}y(x)&=F(x)-L^{-1}\left(\frac{p_2}{s^2+p_1s}L[y]\right)\\&-L^{-1}\left(\frac{p_3}{s^2+p_1s}L[f(y)]\right)\end{aligned} \quad (17)$$

where $F(x)$ represents the term arising from the source term and the prescribed initial conditions.

Now, we apply homotopy perturbation method [35],

$$y(x)=\sum_{n=0}^{\infty}p^ny_n(x) \quad (18)$$

where the term y_n are to recursively calculated and the nonlinear term $f(y)$ can be decomposed as

$$f(y)=\sum_{n=0}^{\infty}p^nH_n(y) \quad (19)$$

for some He's polynomial H_n (see [[21,22] and their referencein]) that are given by

$$\begin{aligned}&H_n(y_0,y_1,y_2,\dots\dots,y_n)\\&=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[f\left(\sum_{i=0}^{\infty}p^iy_i\right)\right]_{p=0}\qquad n=0,1,2,3\dots\dots\end{aligned}$$

Substituting Eqs.(18) and (19) in (17), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}p^ny_n(x)\\&=F(x)-p\left(\begin{aligned}&L^{-1}\left\{\frac{p_2}{(s^2+p_1s)}L\left[\sum_{n=0}^{\infty}p^ny_n(x)\right]\right\}\\&+L^{-1}\left\{\frac{p_3}{(s^2+p_1s)}L\left[\sum_{n=0}^{\infty}p^nH_n(y)\right]\right\}\end{aligned}\right)\end{aligned} \quad (20)$$

which is the coupling of the Laplace transformation and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained:

$$p^0: \quad y_0(x) = F(x),$$

$$p^1: \quad y_1(x) = -\left(\begin{array}{l} L^{-1}\left\{ \dfrac{p_2}{(s^2+p_1 s)} L[y_0(x)] \right\} \\ +L^{-1}\left\{ \dfrac{p_3}{(s^2+p_1 s)} L[H_0(y)] \right\} \end{array} \right)$$

$$p^2: \quad y_2(x) = -\left(\begin{array}{l} L^{-1}\left\{ \dfrac{p_2}{(s^2+p_1 s)} L[y_1(x)] \right\} \\ +L^{-1}\left\{ \dfrac{p_3}{(s^2+p_1 s)} L[H_1(y)] \right\} \end{array} \right) \quad (21)$$

$$p^3: \quad y_3(x) = -\left(\begin{array}{l} L^{-1}\left\{ \dfrac{p_2}{(s^2+p_1 s)} L[y_2(x)] \right\} \\ +L^{-1}\left\{ \dfrac{p_3}{(s^2+p_1 s)} L[H_2(y)] \right\} \end{array} \right).$$

4. Numerical Examples

In this section, two examples are used to illustrate performance of the He-Laplace method. For the sake of comparison with other methods, we consider the same examples as in [6,24,34]. In the process of Computation, all the symbolic and numerical computations are performed by using Mathematica.

Example 4.1. Consider the linear two-point boundary value problem of the form [24,34]

$$y'' = y + \cos(x), \quad 0 < x < 1, \quad (22)$$

Subject to the boundary conditions

$$y(0) = 1 \;, \quad y(1) = 1 \quad (23)$$

The exact solution of the problem is

$$y(x) = \frac{-3\cosh(1) + 3\sinh(1) + \cos(1) + 2}{4\sinh(1)}\exp(x) + \frac{3\cosh(1) + 3\sinh(1) - \cos(1) - 2}{4\sinh(1)}\exp(-x) - \frac{\cos(x)}{2} \quad (24)$$

By applying the aforesaid method subject to the initial condition, we have

$$y(s) = \frac{s+\beta}{s^2} + \frac{1}{s(s^2+1)} + \frac{1}{s^2}L[y] \quad (25)$$

where $y'(0) = \beta$.

The inverse of the Laplace transform implies that

$$y(x) = 1 + \beta x + 1 - \cos x + L^{-1}\left[\frac{1}{s^2}L[y]\right] \quad (26)$$

Now, we apply the homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x) = 2 + \beta x - \cos x + p\left(L^{-1}\left[\frac{1}{s^2}L[y]\right]\right) \quad (27)$$

Comparing the coefficient of like powers of p, we have

$$p^0: \quad y_0(x) = 2 + \beta x - \cos x \,,$$

$$\begin{aligned} p^1: \quad y_1(x) &= L^{-1}\left[\frac{1}{s^2}L[y_0(x)]\right] \\ &= L^{-1}\left[\frac{1}{s^3} + \frac{\beta}{s^4} + \frac{1}{s^3(s^2+1)}\right] \\ &= x^2 + \frac{\beta x^3}{6} - 1 + \cos x, \end{aligned} \quad (28)$$

$$\begin{aligned} p^2: \quad y_2(x) &= L^{-1}\left[\frac{1}{s^2}L[y_1(x)]\right] \\ &= -\cos x + \frac{\beta x^5}{120} + \frac{x^4}{12} - \frac{x^2}{2} + 1, \end{aligned}$$

Proceeding in a similar manner, we have

$$p^3: \quad y_3(x) = \frac{\beta x^7}{5040} + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{360} + \cos x - 1,$$

$$\begin{aligned} p^4: \quad y_4(x) &= \frac{\beta x^9}{362880} - \cos x - \frac{x^2}{2} \\ &\quad + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{20160} + 1, \end{aligned}$$

$$\begin{aligned} p^5: \quad y_5(x) &= \frac{\beta x^{11}}{39916800} + \cos x + \frac{x^2}{2} - \frac{x^4}{24} \\ &\quad + \frac{x^6}{720} - \frac{x^8}{40320} + \frac{x^{10}}{1814400} - 1, \end{aligned}$$

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So that the solution $y(x)$ is given by

$$\begin{aligned} y(x) &= y_0 + y_1 + y_2 + y_3 + y_4 + y_5 \\ &= \beta x + 1 + \frac{\beta x^3}{6} + \frac{\beta x^5}{120} + \frac{x^4}{24} + \frac{\beta x^7}{5040} + \frac{x^6}{360} \\ &\quad + \frac{\beta x^9}{362880} + \frac{x^8}{40320} + \frac{\beta x^{11}}{39916800} + \frac{x^{10}}{1814400}. \end{aligned} \quad (29)$$

The approximation must satisfy the boundary conditions. Imposing the boundary condition of Eq.(23) at x = 1 on Eq.(29), we can obtain the value of the parameter $\beta = -0.8887582849$.

Table 1. Absolute error for example 4.1

x	$\lvert y(x)_{exact} - y_5(x) \rvert$ in EADM	$\lvert y(x)_{exact} - y_5(x) \rvert$ in He-Laplace method
1/8	4.37x10^{-7}	0.2x10^{-9}
2/8	8.07x10^{-7}	0.4x10^{-9}
3/8	1.05x10^{-6}	0.5x10^{-9}
4/8	1.14x10^{-6}	0.6x10^{-9}
5/8	1.06x10^{-6}	0.13x10^{-8}
6/8	8.07x10^{-7}	0.12x10^{-8}
7/8	4.37x10^{-7}	0.11x10^{-8}

Table 1 shows the comparison between exact solution and the series solution obtained by the exponential fitting method (EFM) in [34] and the HLHPM. We compare only five iterations of both the methods. It is found that for the same number of terms, the HLHPM yields relatively more accurate results than other methods. It is obvious that evaluation of more components of u(x) will reasonably improve the accuracy of series solution by using the HLHPM.

Example 4.2. Consider the nonlinear two-point boundary value problem of the form [17,22]

$$u'' = u^2 + 2\pi^2 \cos(2\pi x) - \sin^4(\pi x),\ \ 0 < x < 1, \quad (30)$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = 0. \quad (31)$$

The exact solution of the problem is

$$u(x) = \sin^2(\pi x).$$

By applying the aforesaid method subject to the initial condition, we have

$$\begin{aligned} y(s) &= \frac{s+\beta}{s^2} + \frac{2\pi^2 s}{s^2(s^2+4\pi^2)} \\ &- \frac{24\pi^4}{s^3(s^2+4\pi^2)(s^2+16\pi^2)} + \frac{1}{s^2}L\left[y^2\right] \end{aligned} \quad (32)$$

where $y'(0) = \beta$.

Taking inverse Laplace transform, we have

$$y(x) = L^{-1}\begin{bmatrix} \frac{s+\beta}{s^2} + \frac{2\pi^2}{s(s^2+4\pi^2)} \\ -\frac{24\pi^4}{s^3(s^2+4\pi^2)(s^2+16\pi^2)} + \frac{1}{s^2}L\left[y^2\right] \end{bmatrix}$$

$$\begin{aligned} y(x) &= 1 + \beta x + \frac{1}{2}(1-\cos(2\pi x)) \\ &+ \left[\frac{15 - 24\pi^2 x^2 - 16\cos(2\pi x) + \cos(4\pi x)}{128\pi^2}\right] \\ &+ L^{-1}\left[\frac{1}{s^2}L\left[y^2\right]\right] \end{aligned} \quad (33)$$

Now, we apply homotopy perturbation method [35],

$$y(x) = \sum_{n=0}^{\infty} p^n y_n(x) \quad (34)$$

where the term y_n are to recursively calculated and the nonlinear term f(y) can be decomposed as

$$f(y) = \sum_{n=0}^{\infty} p^n H_n(y) \quad (35)$$

For some He's polynomial H_n that are given by

$$H_n(y_0, y_1, y_2, ...y_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[f\left(\sum_{i=0}^{\infty} p^i y_i\right)\right]_{p=0} \quad (36)$$

$$n = 0,1,2,3,...........$$

$$\sum_{n=0}^{\infty} p^n y_n(x) = f(x) + p\left(L^{-1}\left\{\frac{1}{s^2}L\left[\sum_{n=0}^{\infty} p^n H_n(y)\right]\right\}\right) \quad (37)$$

$$p^0: \quad y_0(x) = f(x)$$

$$p^1: \quad y_1(x) = L^{-1}\left\{\frac{1}{s^2}L\left[H_0(y)\right]\right\}$$

$$p^2: \quad y_2(x) = L^{-1}\left\{\frac{1}{s^2}L\left[H_1(y)\right]\right\}$$

$$p^3: \quad y_3(x) = L^{-1}\left\{\frac{1}{s^2}L\left[H_2(y)\right]\right\}$$

Preceding the steps as in the previous example, we can obtain the value of β and consequently determine the approximate solution with seven iterations $y_7(x)$ .The absolute errors between the exact solution and the results obtained by the shooting method [6], EADM and the HLHPM are compared in Table 2. It is worth to mention that shooting method requires several numerical procedures in obtaining approximations. It is noted that $y_7(x)$ give much better numerical approximations than the results by the EADM. It is interesting to point out that He's polynomial plays an important role in handling nonlinear terms and making the solution procedure in a systematic way.

Table 2. Absolute error for example 4.2

x	$\lvert y(x)_{exact} - y_7(x)\rvert$ in EADM	$\lvert y(x)_{exact} - y_7(x)\rvert$ in He-Laplace method
0.1	$6.9\text{x}10^{-7}$	$-0.10\text{x}10^{-9}$
0.2	$1.3\text{x}10^{-6}$	$0.5\text{x}10^{-9}$
0.3	$1.9\text{x}10^{-6}$	$0.5\text{x}10^{-9}$
0.4	$2.3\text{x}10^{-6}$	$-0.1\text{x}10^{-9}$
0.5	$2.5\text{x}10^{-6}$	$0.1\text{x}10^{-9}$
0.6	$2.3\text{x}10^{-6}$	$0.6\text{x}10^{-9}$
0.7	$1.9\text{x}10^{-6}$	$0.6\text{x}10^{-9}$
0.8	$1.3\text{x}10^{-6}$	$0.7\text{x}10^{-9}$
0.9	$6.9\text{x}10^{-7}$	$0.95\text{x}10^{-9}$

5. Conclusion

In this paper, we have applied He-Laplace method for obtaining approximate solutions of linear and nonlinear second-order two-point boundary value problems. Further, we have compared the performance of the He-Laplace method with extended adomian decomposition method. Numerical examples are presented. On the basis of these examples, for the same number of terms, He-Laplace method yields much better results than the other methods.

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