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Some Identities Involving Common Factors of k-Fibonacci and k-Lucas Numbers

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Abstract Fibonacci sequence stands as a kind of super sequences with fabulous properties. In this paper we present identities involving Common factors of k-Fibonacci and k-Lucas number. Also Binet's formula will employ to this identity.

Keywords: *Fibonacci number, Fibonacci sequence, k- Fibonacci number and k- Lucas number*

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1. Introduction

The Fibonacci sequence and the Lucas sequence are the two shining stars in the vast array of integer sequences. They have fascinated both amateurs and professional mathematicians for centuries, and they continue to charm us with their beauty, abundant applications, and ubiquitous habit of occurring in totally surprising and unrelated places.

The Fibonacci numbers $\{F_n\}_{n\in N}$ are terms of the sequence {0, 1, 1, 2, 3, 5, 8, 13, 21,. ..}, where each term is the sum of the two previous terms, beginning with the values $F_0 = 0$ *and* $F_1 = 1$. The Lucas numbers $\{L_n\}_{n\in N}$ are terms of the sequence {2, 1, 3, 4, 7, 11, 18, 29, 47,...}, where each term is the sum of the two previous terms, beginning with the values $L_0 = 2$ *and* $L_1 = 1$. Many kinds of generalizations of Fibonacci numbers and Lucas numbers have been presented in the literature [6,7,9]. One example of this recent generalization is k-Fibonacci numbers [2] and k-Lucas numbers [4], which are defined by the recurrence relation, depend on one real parameter k. Falcon and Plaza [3] defined k- Fibonacci hyperbolic functions and deduced some properties of k- Fibonacci hyperbolic functions related with the analogous identities for the k- Fibonacci numbers. Falcon [4] studied the k-Lucas numbers and proved various properties related with the k- Fibonacci numbers. Thongmoon [11] and Hoggatt [5] defined various identities for the Fibonacci and Lucas numbers. Gupta and Panwar [10], present generalized identities involving common factors of generalized Fibonacci, Jacobsthal and jacobsthal-Lucas numbers and Binet's formula will employ to obtain the identities. There are a lot of identities of Fibonacci and Lucas numbers described in [1,8]. In this paper, we present generalized identities involving common factors of k- Fibonacci, and k-Lucas numbers.

2. The K-Fibonacci and the K-Lucas Numbers

Before presenting our main results, we will need to introduce some known results and notations.

The k-Fibonacci numbers have been defined in [2] for any real k as follows:

For any positive real number k, the k-Fibonacci sequence say $\{F_{k,n}\}_{n\in N}$ is define recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \quad for\ n \geq 1; \tag{1}$$

With initial condition

$$F_{k,0} = 0,\ \ F_{k,1} = 1 \tag{2}$$

Particular cases of the k-Fibonacci sequence are

If k = 1, the classical Fibonacci sequence is obtained

$$F_0 = 0, F_1 = 1 and\ F_{n+1} = F_n + F_{n-1}\ for\ \ n \geq 1;$$
$$\{F_{k,n}\}_{n\in N} = \{0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ .....\}$$

If k = 2, the classical Pell sequence is obtained

$$P_0 = 0, P_1 = 1 and P_{n+1} = 2P_n + P_{n-1}\ for\ \ n \geq 1;$$
$$\{P_{k,n}\}_{n\in N} = \{0, 1, 2, 12, 29, 70,\ .....\}$$

If k = 3, the following sequence is obtained

$$H_0 = 0, H_1 = 1 and\ H_{n+1} = 2H_n + H_{n-1}\ \ for\ \ n \geq 1\ ;$$
$$\{H_{k,n}\}_{n\in N} = \{0, 1, 3, 10, 33, 109,\ .....\}$$

Binet formula for the nth k-Fibonacci numbers is given by

$$F_{k.n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \tag{3}$$

where r_1 , r_2 are the roots of the characteristic equation $r^2 = kr + 1$ and $r_1 > r_2$; $r_1 = \frac{k + \sqrt{k^2+4}}{2}$, $r_2 = \frac{k - \sqrt{k^2+4}}{2}$ which gives

$$r_1 + r_2 = k, r_1 r_2 = -1, r_1 - r_2 = \sqrt{k^2+4} \tag{4}$$

Similarly for any positive real number k, the k-Lucas sequence [4] say $\{L_{k,n}\}_{n\in N}$ is define recurrently by

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1} \qquad for \;\; n \geq 1 \tag{5}$$

with initial condition

$$L_{k,0} = 2, \;\; L_{k,1} = k \; . \tag{6}$$

Particular cases of the k-Lucas sequence are

If k = 1, the classical Lucas sequence is obtained

$$L_0 = 2, L_1 = 1 \; and \; L_{n+1} = L_n + L_{n-1} \; for \;\; n \geq 1 \, ;$$
$$\{L_{k,n}\}_{n\in N} = \{2, 1, 3, 4, 7, 11, 18, ...\}.$$

If k = 2, the classical Pell- Lucas sequence is obtained

$$p_0 = 2, \quad p_1 = 2 \; and \; p_{n+1} = 2p_n + p_{n-1} \; for \;\; n \geq 1;$$
$$\{p_{k,n}\}_{n\in N} \{2, 2, 6, 14, 34, 82, ...\} \; .$$

If k = 3, the following sequence is obtained

$$H_0 = 2, H_1 = 3 \; and \; H_{n+1} = 2H_n + H_{n-1} \; for \;\; n \geq 1;$$
$$\{H_{k,n}\}_{n\in N} = \{2, 3, 11, 36, 119, 393, ...\}.$$

Binet formula for the nth k-Lucas numbers is given by

$$L_{k.n} = r_1^n + r_2^n \tag{7}$$

where r_1 , r_2 are the roots of the characteristic equation which are given in equation (4).

There are a lot of identities for the common factor of Fibonacci and the Lucas numbers define in [12].

3. Identities for The Common Factors of K-Fibonacci and K-Lucas Number

In this section we present generalized identities involving common factors of k-Fibonacci and k-Lucas numbers. We shall use the Binet's formula for the k-Fibonacci numbers and k-Lucas numbers for derivation.

3.1. Proposition

$F_{k.4n+p} - F_{k,p} = F_{k,2n} L_{k,2n+p}$, *where* $n \geq 1$ *and* $p \geq 0$.

Proof

$$\begin{aligned} F_{k,2n} L_{k,2n+p} &= \left(\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2} \right) \left(r_1^{2n+p} + r_2^{2n+p} \right) \\ &= \left(\frac{r_1^{4n+p} - r_2^{4n+p}}{r_1 - r_2} \right) - \left(\frac{r_2^{2n} r_1^{2n+p} - r_1^{2n} r_2^{2n+p}}{r_1 - r_2} \right) \\ &= F_{k,4n+p} - \frac{1}{r_1 - r_2} \left[(-1)^{2n} r_1^p - (-1)^{2n} r_2^p \right] \\ &= F_{k,4n+p} - F_{k,p} \end{aligned} \tag{8}$$

3.2. Proposition

$F_{k.4n+p} + F_{k,p} = F_{k,2n+p} L_{k,2n}$, *where* $n \geq 1$ *and* $p \geq 0$.

Proof

$$\begin{aligned} F_{k,2n+p} L_{k,2n} &= \left(\frac{r_1^{2n+p} - r_2^{2n+p}}{r_1 - r_2} \right) \left(r_1^{2n} + r_2^{2n} \right) \\ &= \left(\frac{r_1^{4n+p} - r_2^{4n+p}}{r_1 - r_2} \right) + \left(\frac{r_2^{2n} r_1^{2n+p} - r_1^{2n} r_2^{2n+p}}{r_1 - r_2} \right) \\ &= F_{k,4n+p} + \frac{1}{r_1 - r_2} \left[(-1)^{2n} r_1^p - (-1)^{2n} r_2^p \right] \\ &= F_{k,4n+p} + F_{k,p} \end{aligned} \tag{9}$$

3.3. Proposition

$L_{k.4n+p} - L_{k,p} = (r_1 - r_2)^2 F_{k,2n} F_{k,2n+p}$, *where* $n \geq 1$ *and* $p \geq 0$.

Proof

$$\begin{aligned} &(r_1 - r_2)^2 F_{k,2n} F_{k,2n+p} \\ &= (r_1 - r_2)^2 \left(\frac{r_1^{2n} - r_2^{2n}}{r_1 - r_2} \right) \left(\frac{r_1^{2n+p} - r_2^{2n+p}}{r_1 - r_2} \right) \\ &= (r_1^{4n+p} + r_2^{4n+p}) - (r_1 r_2)^{2n} (r_1^p + r_2^p) \\ &= L_{k,4n+p} - L_{k,p} \end{aligned} \tag{10}$$

3.4. Proposition

$L_{k.4n+p} + L_{k,p} = L_{k,2n} L_{k,2n+p}$, *where* $n \geq 1$ *and* $p \geq 0$.

Proof

$$\begin{aligned} L_{k,2n} L_{k,2n+p} &= (r_1^{2n} + r_2^{2n})(r_1^{2n+p} + r_2^{2n+p}) \square \\ &= (r_1^{4n+p} + r_2^{4n+p}) + (r_1 r_2)^{2n} (r_1^p + r_2^p) \\ &= L_{k,4n+p} + L_{k,p} \end{aligned} \tag{11}$$

3.5. Proposition

$F_{k.4n+p} - F_{k,p} = F_{k,n} L_{k,n} L_{k,2n+p}$, *where* $n \geq 1$ *and* $p \geq 0$.

Proof

$$\begin{aligned} F_{k,n} L_{k,n} L_{k,2n+p} &= \left(\frac{r_1^n - r_2^n}{r_1 - r_2} \right) \left(r_1^n + r_2^n \right) \left(r_1^{2n+p} + r_2^{2n+p} \right) \\ &= \left(\frac{r_1^n - r_2^n}{r_1 - r_2} \right) \left(r_1^{3n+p} + r_2^{3n+p} + r_1^n r_2^{2n+p} + r_2^n r_1^{2n+p} \right) \\ &= \left(\frac{r_1^{4n+p} - r_2^{4n+p}}{r_1 - r_2} \right) - \left(\frac{r_1^p - r_2^p}{r_1 - r_2} \right) \\ &= F_{k.4n+p} - F_{k,p} \end{aligned} \tag{12}$$

3.6. Proposition

$L_{k.4n+p} - L_{k,p} = (r_1 - r_2)^2 F_{k,n} L_{k,n} F_{k,2n+p}$,

where $n \geq 1$ *and* $p \geq 0$.

Proof

$$
\begin{aligned}
&(r_1-r_2)^2 F_{k,n}L_{k,n}F_{k,2n+p}\\
&=(r_1-r_2)^2\left(\frac{r_1^n-r_2^n}{r_1-r_2}\right)\left(r_1^n+r_2^n\right)\left(\frac{r_1^{2n+p}-r_2^{2n+p}}{r_1-r_2}\right)\\
&=\left(r_1^n-r_2^n\right)\left(r_1^{3n+p}-r_2^{3n+p}-r_1^n r_2^{2n+p}+r_2^n r_1^{2n+p}\right) \quad (13)\\
&=\left(r_1^{4n+p}+r_2^{4n+p}\right)-\left(r_1^p+r_2^p\right)\\
&=L_{k.4n+p}-L_{k,p}
\end{aligned}
$$

3.7. Proposition

$F_{k.3n+p}-(-1)^n F_{k,n+p}=F_{k,n}L_{k,2n+p}$, *where* $n\geq 1$ *and* $p\geq 0$.

Proof

$$
\begin{aligned}
F_{k,n}L_{k,2n+p}&=\left(\frac{r_1^n-r_2^n}{r_1-r_2}\right)\left(r_1^{2n+p}+r_2^{2n+p}\right)\\
&=\left(\frac{r_1^{3n+p}-r_2^{3n+p}}{r_1-r_2}\right)-\left(\frac{r_2^n r_1^{2n+p}-r_1^n r_2^{2n+p}}{r_1-r_2}\right) \quad (14)\\
&=F_{k,3n+p}-\frac{(-1)^n}{r_1-r_2}\left(r_1^{n+p}-r_2^{n+P}\right)\\
&=F_{k.3n+p}-(-1)^n F_{k,n+p}
\end{aligned}
$$

In [13], if we take $i=n$ *and* $n=n+p$ in Lemma 1(i), then we have the following proposition.

3.8. Proposition

$F_{k.3n+p}+(-1)^n F_{k,n+p}=F_{k,2n+p}L_{k,n}$, *where* $n\geq 1$ *and* $p\geq 0$.

Proof

$$
\begin{aligned}
F_{k,2n+p}L_{k,n}&=\left(\frac{r_1^{2n+P}-r_2^{2n+p}}{r_1-r_2}\right)\left(r_1^n+r_2^n\right)\\
&=\left(\frac{r_1^{3n+p}-r_2^{3n+p}}{r_1-r_2}\right)+\left(\frac{r_1^{2n+p}r_2^n-r_2^{2n+p}r_1^n}{r_1-r_2}\right) \quad (15)\\
&=F_{k,3n+p}+\frac{(-1)^n}{r_1-r_2}\left(r_1^{n+p}-r_2^{n+P}\right)\\
&=F_{k.3n+p}+(-1)^n F_{k,n+p}
\end{aligned}
$$

3.9. Proposition

$L_{k.3n+p}-(-1)^n L_{k,n+p}=(r_1-r_2)^2 F_{k,n}F_{k,2n+p}$,

where $n\geq 1$ *and* $p\geq 0$.

Proof

$$
\begin{aligned}
&(r_1-r_2)^2 F_{k,n}F_{k,2n+p}\\
&=(r_1-r_2)^2\left(\frac{r_1^n-r_2^n}{r_1-r_2}\right)\left(\frac{r_1^{2n+p}-r_2^{2n+p}}{r_1-r_2}\right)\\
&=(r_1^{3n+p}+r_2^{3n+p})-(r_2^n r_1^{2n+p}+r_1^n r_2^{2n+p}) \quad (16)\\
&=L_{k,3n+p}-(-1)^n\left(r_1^{n+p}+r_2^{n+P}\right)\\
&=L_{k,3n+p}-(-1)^n L_{k,n+p}
\end{aligned}
$$

In [13], if we take $i=n$ *and* $ni=2n+p$ in Lemma 1(ii), then we have the following proposition.

3.10. Proposition

$L_{k.3n+p}+(-1)^n L_{k,n+p}=L_{k,n}L_{k,2n+p}$, *where* $n\geq 1$ *and* $p\geq 0$.

Proof

$$
\begin{aligned}
L_{k,n}L_{k,2n+p}&=\left(r_1^n+r_2^n\right)\left(r_1^{2n+p}+r_2^{2n+p}\right)\\
&=(r_1^{3n+p}+r_2^{3n+p})+(r_2^n r_1^{2n+p}+r_1^n r_2^{2n+p})\\
&=L_{k,3n+p}+(-1)^n\left(r_1^{n+p}+r_2^{n+P}\right) \quad (17)\\
&=L_{k,3n+p}+(-1)^n L_{k,n+p}
\end{aligned}
$$

For different values of p, Proposition (3.1) to Proposition (3.10) can be expressed for even and odd k-Fibonacci and k-Lucas numbers.

Identities (8) to (17) give a relation between k-Fibonacci and k-Lucas numbers by using their subscripts.

4. Conclusion

In this paper we present generalized identities involving common factors of k- Fibonacci and k-Lucas numbers. Mainly Binet's formula employ for the identities. The concept can be executed for generalized second order recursive sequences as well as polynomials.

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