*American Journal of Mathematical Analysis, 2014, Vol. 2, No. 2, 28-32*
Available online at http://pubs.sciepub.com/ajma/2/2/3

DOI:10.12691/ajma-2-2-3

# Exact Soliton Solutions of the 1D Generalized Gross-Pitaevskii Equation with Quadratic Potential and Parameterized Nonlinearity

**Ying Wang, Qing-Chun Zhou, Ya-Lun Li, Yu Zhou***

School of Mathematics and Physics, Jiangsu University of Science and Technology, Jiangsu, China
*Corresponding author: yzhou@just.edu.cn

*Received May 06, 2014; Revised May 26, 2014; Accepted June 11, 2014*

**Abstract** We investigate the 1D generalized Gross-Pitaevskii equation (GGPE) with quadratic potential and parameterized nonlinearity. The coefficients of terms of GGPE studied are arbitrary functions of time *t*. The exact solution(s) of the GGPE are obtained via expansion method with particular soliton features highlighted.

***Keywords:*** *Gross-Pitaevskii equation, lens-type transformation, expansion method, soliton*

**Cite This Article:** Ying Wang, Qing-Chun Zhou, Ya-Lun Li, and Yu Zhou, "Exact Soliton Solutions of the 1D Generalized Gross-Pitaevskii Equation with Quadratic Potential and Parameterized Nonlinearity." *American Journal of Mathematical Analysis*, vol. 2, no. 2 (2014): 28-32. doi: 10.12691/ajma-2-2-3.

## 1. Introduction

It is well known that in reality many systems behave nonlinearly and described by nonlinear partial differential equations. Among these is the nonlinear Schrodinger equation that has been applied extensively and play principal role in modeling so many phenomena like the functioning of optical fibre and the realization of Bose-Einstein condensation (BEC). Due to the particular feature that the nonlinear interaction could be tuned flexibly via Feshbach resonance, BEC attracts a lot of attention in the study of nonlinear dynamics like controllable bright/dark soliton features and Gross-Pitaevskii equation (GPE) has been successfully applied to this particular application scenario and well investigated [1-8].

But in order to fully incorporate the mechanism of tunable Feshbach resonance, the generalized Gross-Pitaevskii equation (GGPE) with nonlinearity of arbitrary power index of modulus of wave function is more appropriate [9,10]. The investigation of such GGPE with parameterized nonlinearity however is relatively rare. In this paper, we will investigate the model for the system of BEC with tunable Feshbach resonance control in the cigar-shaped harmonic trap and gravitational field that is described by the 1D GGPE. To give a more thorough and general treatment, we allow the coefficients of external(quadratic) potential and nonlinear terms to be arbitrary functions of t. We demonstrate that such 1D GGPE with parameterized nonlinearity and quadratic external potential is analytically solvable and obtain its exact solution through modified ($1/G$, $G'/G$) expansion method [11].

The paper is arranged as follows, in the next section, we give the outline of solution finding strategies and main results, followed by the commentary explanation and discussion, with the concluding and summarization statement for the work presented given in the last section.

## 2. Methodologies

### 2.1. Problem Formulation

The dimensionless 1D generalized Gross-Pitaevskii equation (GGPE) to be studied reads

$$\begin{aligned} i\frac{\partial \psi(x,t)}{\partial t} &= -\frac{\partial^2}{\partial x^2}\psi(x,t) + k_2(t)x^2\psi(x,t) \\ &+ k_1(t)x\psi(x,t) + g(t)|\psi(x,t)|^{2\gamma}\psi(x,t) \end{aligned} \tag{1}$$

Where $\gamma$ is a real constant power index in the nonlinear term determined by the experiment that is discussed in Refs. [12-17]. The first term on the right hand side of (1) is dispersion term, the second term on the RHS is arising from external harmonic magnetic trapping potential, the third term is from gravitational potential, the fourth term is the interaction term with Landau coefficient $g(t) > 0$ corresponding to the repulsive interaction, $g(t) < 0$ corresponding to the attractive interaction. The GGPE (1) is the reduction from its 3D analog in certain scenario like cigar-shaped elongated external harmonic potential $V(r) = \frac{1}{2}m(\omega_\rho^2\rho^2 + \omega_x^2 x^2)$ with $\omega_\rho >> \omega_x$, similar as what were discussed in some prior work[18, 19], $k_2(t), k_1(t)$ are chosen to be arbitrary function of time t.

To eliminate the interdependence between $k(t)$ and $g(t)$ to make the 1D GGPE analytically solvable, we introduce a parameter function $\sigma(t)$ into the coefficients of the above equation through the following transformation

$$x' = \sigma(t')x + \sigma_2(t'), \quad (2)$$

$$t' = t, \quad (3)$$

$$\psi(x,t) = \sigma^{1/2}(t') \times \exp[i(\frac{1}{2}\frac{\sigma_{t'}(t')}{\sigma(t')}x^2 + \sigma_3(t)x + \sigma_4(t))]\phi(x',t') \quad (4)$$

Formula (2), (3) and (4) can be thought as modified lens-type transformation[20,21]. We could express $\sigma_2(t)$, $\sigma_3(t)$, $\sigma_4(t)$ using $\sigma(t)$ with the following three constraint formula

$$\sigma_2(t) = \frac{k_1(t) - [\sigma_{2'}(t)/\sigma_2(t)]'}{2\sigma(t)[k_2(t)/\sigma^2(t) - 1/4\sigma_t^2(t)/\sigma^2(t) - 1/4(\sigma_t(t)/\sigma(t))']} \quad (5)$$

$$\sigma_3(t) = -\sigma_{2'}(t)/2\sigma(t) \quad (6)$$

$$\sigma_{4'}(t) = -1/4\sigma_{2'}^2(t)/\sigma^2(t) + \frac{[k_1(t) - 1/2(\sigma_{2'}(t)/\sigma(t))']^2}{4\sigma(t)[k_2(t)/\sigma^2(t) - 1/4\sigma_t^2(t)/\sigma^2(t) - 1/4(\sigma_t(t)/\sigma(t))']} \quad (7)$$

with $\sigma(t)$ to be determined later. Substituting (2), (3) and (4) into Eq.(1) and changing notation from $(x',t')$ to $(x,t)$ we get the generalized GP equation with modified coefficients as

$$i\phi_t + \sigma^2(t)\phi_{xx} + [\frac{k(t)}{\sigma^2(t)} - (\frac{\sigma_t(t)}{2\sigma(t)})^2 - \frac{1}{4}(\frac{\sigma_t(t)}{\sigma(t)})_t]x^2\phi + g(t)\sigma^\gamma(t)|\phi|^{2\gamma}\ \varphi = 0 \quad (8)$$

Assume wavefunction is of the form

$$\phi(x,t) = v^{1/\gamma}(x,t)e^{i\theta(x,t)} \quad (9)$$

substitute (9) into Eq.(8), we get the following equations for $v(x,t)$ and $\theta(x,t)$

$$v^2\theta_t + \sigma^2(t)(a_0 vv_{xx} + b_0 v_x^2 + v^2\theta_x^2) + \alpha(t)x^2v^2 + \beta(t)v^4 = 0 \quad (10)$$

$$v_t + \sigma^2(t)(2v_x\theta_x + \gamma v\theta_{xx}) = 0 \quad (11)$$

where $\alpha(t) = k(t)/\sigma^2(t) - 1/4(\sigma_t(t)/\sigma(t))^2 - (\sigma_t(t)/4\sigma(t))_t$ , $\beta(t) = g(t)\sigma^\gamma(t)/\hbar\omega, a_0 = -1/\gamma$ ,and $b_0 = -(1-\gamma)/\gamma$ are constants. In the ensuing equations solving steps we will work on Eqs.(10-11) for the final solution of the GGPE.

## 2.2. Method Outline

In order to obtain the exact analytical solution of Eqs.(2.1), we consider the ( $G'/G$ , $1/G$ )-expansion method, which is applicable to solving the nonlinear partial differential equation of the form

$$F(u, u_t, u_x, u_{xx}, \ldots) = 0 \quad (12)$$

Where $F$ is the polynomial of unknown function $u(x,t)$ and its partial derivatives of various order. The basic idea of ( $G'/G$ , $1/G$ )-expansion is trying to express the unknown function as polynomial of $G'(\xi)/G(\xi)$ and $1/G$, with $G(\xi)$ defined as function of $\xi = p(t)x + q(t)$ like

$$\frac{d^2}{d\xi^2}G(\xi) + \lambda G(\xi) = 0 \quad (13)$$

where $\lambda$ is constant. There are three cases for the solution of Eq.(13):

*Case 1* When $\lambda < 0$,

$$G(\xi) = A_1 \sinh\sqrt{-\lambda}\xi + A_2 \cosh\sqrt{-\lambda}\xi, \quad (14)$$

and we have

$$1/G^2 = -\frac{1}{\lambda\rho}[(G'/G)^2 + \lambda] \quad (15)$$

where $A_1$ and $A_2$ are two arbitrary constants and $\rho = A_1^2 - A_2^2$.

*Case 2* When $\lambda > 0$,

$$G(\xi) = A_1 \sin\sqrt{\lambda}\xi + A_2 \cos\sqrt{\lambda}\xi, \quad (16)$$

and we have

$$1/G^2 = \frac{1}{\lambda\rho}[(G'/G)^2 + \lambda] \quad (17)$$

where $A_1$ and $A_2$ are two arbitrary constants and $\rho = A_1^2 + A_2^2$.

*Case 3* When $\lambda = 0$, the general solution of Eq.(13) is

$$G(\xi) = A_1\xi + A_2, \quad (18)$$

and we have

$$1/G^2 = \frac{1}{A_1^2}[(G'/G)^2] \quad (19)$$

where $A_1$ and $A_2$ are two arbitrary constants. From (15,17,19), we can express the the solution $u(x,t)$ of (12) as

$$u(x,t) = \sum_{i=0}^{m} z_i(t)(\frac{G'(\xi)}{G\xi})^i + \sum_{i=1}^{m} h_i(t)\frac{1}{G(\xi)}(\frac{G'(\xi)}{G\xi})^{i-1}, \quad h_m^2(t) + z_m^2(t) \neq 0 \quad (20)$$

Substituting (20) into the original nonlinear partial differential equation (12) and making use of relations (13) and (15,17,19), $m$ can be set by balancing between the highest differential term and nonlinear term and express $F$ in (12) as a polynomial of $G'(\xi)/G(\xi)$ plus another polynomial of $G'(\xi)/G(\xi)$ times $1/G(\xi)$. The Eq. (12) is solved by setting the coefficients of all terms ($G'^i(\xi)/G^i(\xi)$ and $G'^j(\xi)/G^j(\xi)1/G(\xi)$) of $F$ to zero. This will result in a set of over determined ODEs for $h_i(t)$ and $z_j(t)$, that will put the unknown function $u(x,t)$ shown in (20) in a explicit form if the ODEs are solved consistently.

### 2.3. Procedure Details and Results

To proceed with the ($G'/G$, $1/G$)-expansion problem solving strategy for Eq.(2.1), we assume $v(x,t)$ and $\theta(x,t)$ in Eq.(2.1) are of the form

$$v(x,t)=\sum_{i=0}^{m} z_i(t)\left(\frac{G'(\xi)}{G\xi}\right)^i+\sum_{i=1}^{m} h_i(t)\frac{1}{G(\xi)}\left(\frac{G'(\xi)}{G\xi}\right)^{i-1}, \quad h_m^2(t)+z_m^2(t)\neq 0 \qquad (21)$$

$$\theta(x,t)=\Phi(t)x^2+\Gamma(t)x+\Omega(t) \qquad (22)$$

with $\xi$ and $G(\xi)$ defined as that in (13). The highest power index $m$ in (21) is set by balancing between the highest derivative term and the nonlinear term when substituting (21) and (22) into Eq.(2.1) and making use of (13) and (15,17,19). The balancing formula is $2m+2=4m$, so $m=1$; hence $v(x,t)$ in (21) can be written in two possible formats as

$$v(x,t)=h(t)\frac{1}{G(\xi)}+f(t) \qquad (23)$$

$$v(x,t)=z(t)\frac{G'(\xi)}{G(\xi)}+f(t) \qquad (24)$$

The problem solving details for the above two cases are very similar, we show the explicit details for (23). After plugging the formatted solution (21) and (22) into Eq.(10-11), making use of formula (13) and (15,17, 19), we have the equations of (10-11) expressed as polynomials for $x^j(G'(\xi)/G(\xi))^i(1/G(\xi))^r$, where $j=0,1,2$; $i\in[0,4]$, and $r=0$ or 1. Setting the coefficients of every term $x^j(G'(\xi)/G(\xi))^i(1/G(\xi))^r$ to zero, we get the following set of ODEs, (define M=$G'(\xi)/G(\xi)$,N=$1/G(\xi)$).

For Eq.(10), we have

$$x^2M^2:\quad h^2(t)[\Phi'(t)+4\sigma^2(t)\Phi^2(t)+\alpha(t)]=0, \qquad (25)$$

$$x^2N:\quad 2h(t)f(t)[\Phi'(t)+4\sigma^2(t)\Phi^2(t)+\alpha(t)]=0, \qquad (26)$$

$$x^2:(f^2(t)-h^2(t)/\rho)[\Phi'(t)+4\sigma^2(t)\Phi^2(t)+\alpha(t)]=0 \qquad (27)$$

$$xM^2:\quad h^2(t)[\Gamma'(t)+4\sigma^2(t)\Phi(t)\Gamma(t)]=0, \qquad (28)$$

$$xN:\quad 2h(t)f(t)[\Gamma'(t)+4\sigma^2(t)\Phi(t)\Gamma(t)]=0, \qquad (29)$$

$$x:(f^2(t)-h^2(t)/\rho)[\Gamma'(t)+4\sigma^2(t)\Phi(t)\Gamma(t)]=0 \qquad (30)$$

and

$$M^4:h^2(t)[2a_0\sigma^2(t)p^2(t)+b_0\sigma^2(t)p^2(t)-\beta(t)h^2(t)/(\lambda\rho)]=0 \qquad (31)$$

$$M^2N:\quad \beta(t)f(t)h^3(t)/(\lambda\rho)=0, \qquad (32)$$

$$M^2:h^2(t)[\Omega'(t)+\sigma^2(t)\Gamma(t)^2+\sigma^2(t)p^2(t)(3a_0+b_0)+6\beta(t)f^2(t)]=0, \qquad (33)$$

$$N:f(t)h(t)[\Omega'(t)+\sigma^2(t)\Gamma(t)^2+\beta(t)(f^2(t)-h^2(t)/\rho)]=0, \qquad (34)$$

$$M^0:\quad (f^2(t)-h^2(t)/\rho)(\Omega'(t)+\sigma^2(t)\Gamma(t)^2)+a_0\frac{\lambda}{\rho}\sigma^2(t)h^2(t)p^2(t)+\beta(t)[(f^2(t)-h^2(t)/\rho)^2-4h^2(t)f^2(t)/\rho]=0, \qquad (35)$$

While for Eq.(11) we have

$$xMN:h(t)[p'(t)+4\sigma^2(t)\Phi(t)p(t)]=0 \qquad (36)$$

and

$$MN:\quad h(t)[q'(t)+4\sigma^2(t)p(t)\Gamma(t)]=0, \qquad (37)$$

$$N:\quad h'(t)+2\gamma\sigma^2(t)\Phi(t)h(t)=0, \qquad (38)$$

$$M^0:\quad f'(t)+2\gamma\sigma^2(t)\Phi(t)f(t)=0, \qquad (39)$$

From the above ODEs (25)-(39), we can see by introducing new parameter function $\sigma(t)$, both $k_i(t)$ (in $\alpha(t)$) and $g(t)$ (in $\beta(t)$) can change freely.

The above ODEs can be solved analytically, with appropriate boundary conditions, and the solutions could be expressed as (We express the unknown functions of $t$ in $\Phi(t)$ and $\sigma(t)$ whose analytical forms are analyzed in the end):

$$p(t)=C_1\exp[-4\int\sigma^2(s)\Phi(s)ds] \qquad (40)$$

$$q(t)=-2C_1C_2\int[\sigma^2(s)\cdot\exp(-8\int\sigma^2(z)\Phi(z)dz)]ds \qquad (41)$$

$$\Gamma(t)=C_2\exp[-4\int\sigma^2(s)\Phi(s)ds] \qquad (42)$$

$$\Omega(t)=-[(3a_0+b_0)C_1^2+C_2^2]\times\int[\sigma^2(s)\cdot\exp(-8\int\sigma^2(z)\Phi(z)dz)ds] \qquad (43)$$

$$h(t)=C_3\exp[-2\gamma\int\sigma^2(s)\Phi(s)ds] \qquad (44)$$

$$f(t)=0 \qquad (45)$$

$$G(\xi)=\sqrt{-\rho}\cosh(\sqrt{-\lambda}(\xi-\xi_0)),\quad \lambda<0, g(t)<0 \qquad (46)$$

$$G(\xi)=\sqrt{\rho}(\xi-\xi_0), \lambda=0, g(t)<0 \qquad (47)$$

$$G(\xi)=\sqrt{-\rho}\cos(\sqrt{\lambda}(\xi-\xi_0)), \lambda>0, g(t)>0 \qquad (48)$$

$$v(x,t)=\sqrt{-\rho}h(t)\operatorname{sech}[\sqrt{-\lambda}(p(t)x+q(t)-\xi_0)],\quad \lambda<0, g(t)<0 \qquad (49)$$

$$v(x,t)=\sqrt{\rho}(p(t)x+q(t)-\xi_0), \lambda=0, g(t)<0 \qquad (50)$$

$$v(x,t)=\sqrt{-\rho}h(t)\sec[\sqrt{\lambda}(p(t)x+q(t)-\xi_0)],\quad \lambda>0, g(t)>0 \qquad (51)$$

Where $C_1$, $C_2$, $\lambda$, $\rho$ and $\xi_0$ are arbitrary constants, $\sigma(t)$ and $\Phi(t)$ are determined by the following pair of equations:

$$\sigma'(t)+\frac{g'(t)}{(2-\gamma)g(t)}\sigma(t)-4\sigma^3(t)\Phi(t)=0, \qquad (52a)$$

$$\Phi'(t)+4\sigma^2(t)\Phi^2(t)+\alpha(t)=0, \qquad (52b)$$

where $g(t)$ and $k_i(t)$ (in $\alpha(t)$ ) are arbitrary functions of $t$ ; however, we always have constants $g(t)=g$ and $k_i(t)=k_i$ in experiments. So $\sigma(t)$ can be directly expressed by $\Phi(t)$ from Eq. (52) as

$$\sigma(t)=\pm(C_4-8\int\Phi(s)ds)^{-1/2} \quad (53)$$

with specific constant $C_4$ . Eq.(53) could also be integrated with constant $k_i$ , $g$ using Maple for example. Since generally $\Phi(t)$ can not be expressed in simple rudimentary function form, we omit the formula that shows explicit the expression for $\Phi(t)$ here. Fortunately, for interested case where both $k_i(t)$ and $g(t)$ are constants, we can bypass the explicit expression for $\Phi(t)$ and reach directly the explicit expressions for the parameters in final solution, as explained in the following section.

# 3. Comments and Discussions

For the solution we obtained in previous section, returning to the original coordinates, from (2),(4),(9) and (49-51), the final solution take the following form

$$|\psi(x,t)|=\begin{cases}\sqrt{\sigma(t)}(-\lambda h^2(t))^{1/2\gamma} \\ \times\operatorname{sech}^{1/\gamma}[\sqrt{-\lambda}\chi(t)+q_0(t)], \lambda<0, g(t)<0 \\ \sqrt{\rho}(x-\sigma_2(t)/\sigma(t))^{1/2\gamma}, \lambda=0, g(t)<0 \\ \sqrt{\sigma(t)}(\lambda h^2(t))^{1/2\gamma} \\ \times\sec^{1/\gamma}[\sqrt{\lambda}\chi(t)+q_0(t)], \lambda>0, g(t)>0\end{cases} \quad (54a)$$

where $\chi(t)=p(t)\sigma(t)(x-\sigma_2(t)/\sigma(t)), q_0(t)=q(t)-\xi_0$, $\sigma(t), h(t), p(t), q(t)$ are given in (40-45). We are particularly interested in the solution of bright soliton type given in first case in (54a) for $\lambda<0$ and $g(t)<0$ which corresponds to the attractive nonlinear interaction. In this case, with $k(t)=k$ , $g(t)=g$ as constants, from the previous results (40-45) and (54), it is not hard to see that $p(t)\sigma(t)=\kappa$ is constant, $q(t)=\varpi t$ ( $\varpi$ is constant), the soliton's shape remain unchanged with time and the peak moves with constant speed $\varpi/\kappa$ perturbed by the gravitational field shown in $\sigma_2(t)/\sigma(t)$ . When $g(t)$ (or $k(t)$ )depends on $t$ , $p(t)\sigma(t)$ , which indicates the compressing ratio of the travelling wave along the $x$ -direction, generally is not constant and the format of $\sigma_2(t)/\sigma(t)-q(t)/(p(t)\sigma(t))$ is complicated, which mean that there is shape variation in the traveling wave and the wave peak move with acceleration.

Also our analysis could be extended to the case of repulsive nonlinear interaction( $g(t)>0$ ) and $\lambda<0$ which corresponds to the solution format (24). In a similar manner as (23), we can obtain a very similar set of ODEs as that of (25-39), and reach an analytical solution of dark soliton type for the GGPE as follow

$$|\psi(x,t)|=\sqrt{\sigma(t)}(-\lambda h^2(t))^{1/2\gamma}\tanh^{1/\gamma}[\sqrt{-\lambda}\chi(t)+q_0(t)] \quad (54b)$$

with $\sigma(t), h(t), p(t), q(t)$ given by the same formula as (40-45). When $k(t)$ and $g(t)$ are constants, $p(t)\sigma(t)$ is constant, the dark soliton shape does not change, also when $g(t)$ or $k(t)$ (or both) has time dependence, generally there will be modulation effects on the shape of traveling wave which normally has non-zero acceleration.

The analysis and results presented to this point should be typical, we could leave for future work the extended study, which include for example more mathematical sound solution format(s) based on expansion method.

# 4. Conclusion

In this paper, we find exact solutions for a much broader category of the generalized 1D Gross-Pitaevskii equation with general quadratic potential. The coefficient functions of the terms of the generalized GPE are arbitrary. We also give a generalized treatment of the nonlinearity with parameterized power index. We eliminate the restriction shown in some previous work on GPE which require a constraint formula connecting variable coefficients of the equation terms. We reach the exact analytical solutions of the GGPE using modified ( $G'/G$ , $1/G$ ) expansion method, and for typical setting for the parameters, we get soliton like traveling wave solutions which are similar to the solutions obtained in previous work on GPE but with general time modulation to the amplitude and shape of the traveling wave arising from the variable coefficients of the equation terms.

# Acknowledgement

This work was supported by NSF-China under Grant Nos. 11105039, 11204274, and 11205071.

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