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On Location of Zeros of Polynomials

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Abstract If $p(z)=\sum_{j=0}^{n}\alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j, j = 0, 1, ….., n are real numbers. In this paper we obtain a generalization of well known result of Eneström -Kakeya concerning the bounds for the moduli of the zeros of polynomials with complex coefficients which improve upon some results due to A. Aziz and Q.G Mohammad and others.

Keywords: *complex polynomials, zeros, Eneström – Kakeya Theorem*

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1. Introduction and Statement of Results

The following result known as Eneström-Kakeya theorem, is well known in the theory of distribution of zeros of polynomials was firstly proved by Eneström [2] and Kakeya [5] and later independently by Hurwitz [3].

Theorem 1.1. If $p(z)=\sum_{j=0}^{n}a_j z^j$ is a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge \dots \ge a_1 \ge a_0 > 0,$$

then all the zeros of $p(z)$ lie in $|z| \le 1$.

Joyal, Labelle and Rahman [4] extended Theorem A to the polynomial whose coefficients are monotonic but not necessarily non-negative by proving the following:

Theorem 1.2. Let $p(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge \dots \ge a_1 \ge a_0,$$

then all the zeros of $p(z)$ lie in

$$|z| \le \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}.$$

Aziz and Zargar [6] relaxed the hypothesis of Eneström-Kakeya theorem and proved the following extension of Theorem 1.2.

Theorem 1.3. Let $p(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$,

$$ka_n \ge a_{n-1} \ge \dots \ge a_1 \ge a_0,$$

then all the zeros of $p(z)$ lie in

$$|z+k-1| \le \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

By using Schwartz lemma, Aziz and Mohammad [1] generalized Eneström -Kakeya theorem in a different way and proved:

Theorem 1.4. If $p(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with real positive coefficients. If $t_1 > t_2 \ge 0$ can be found such that

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0, \quad r = 1,2,\dots,n+1; a_{-1} = a_{n+1} = 0,$$

then all the zeros of $p(z)$ lie in $|z| \le t_1$.

In this paper, we prove some generalizations and extensions of Theorem1.1, Theorem 1.2, Theorem 1.3 and Theorem 1.4. In this direction we first present the following interesting result which is generalization of Theorem 1.4.

Theorem 1.5. If $p(z)=\sum_{j=0}^{n}\alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j, j = 0, 1, ….., n are real numbers and for certain non-negative real numbers t_1, t_2 with $t_1 > t_2$ and $t_1 \ne 0$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0, r = 1,2,\dots,n+1$$
$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \ge 0, \quad r = 1,2,\dots,n+1$$
$$a_{n+1} = 0 = a_{-1} = a_{-2}$$
$$b_{n+1} = 0 = b_{-1} = b_{-2}$$

then all the zeros of $p(z)$ lie in

$$|z| \le \frac{|\alpha_n + M|t_1}{|\alpha_n|},$$

where

$$M = \left(|a_0| - a_0\right)\frac{t_2}{t_1^{n+1}} + \left(|b_0| - b_0\right)\frac{t_2}{t_1^{n+1}}.$$

Remark 1.1. If in Theorem 1.5, we assume that all the coefficients are real and positive, then M=0 and it reduces to Theorem 1.4 due to Aziz and Mohammad [1].

In particular, if we choose t_2=0 and $t_1 \equiv t$ in Theorem 1.5, then we have the following result.

Corollary 1.1. If $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j, j = 0, 1, 2 ….., n are real numbers. If $t > 0$ be such that

$$a_n t^n \ge a_{n-1}t^{n-1} \ge .... \ge a_1 t \ge a_0$$
$$b_n t^n \ge b_{n-1}t^{n-1} \ge .... \ge b_1 t \ge b_0$$

then all the zeros of $p(z)$ lie in

$$|z| \le \frac{|\alpha_n + M'|t}{|\alpha_n|},$$

where

$$M' = \frac{1}{t^n}\left[\left(|a_0| - a_0\right) + \left(|b_0| - b_0\right)\right].$$

Remark 1.2. If we consider all the coefficients in Corollary 1.1 to be real and choose t = 1, then we get Theorem 1.2due to Joyal, Labelle and Rahman [4]. In addition, if we choose all coefficients to be positive, then we obtain the well-known Eneström-Kakeya Theorem (Theorem 1.1).

Next we prove the following result which is also a generalization of Theorem

Theorem.1.6. Let $p(z) = \sum_{j=0}^{n} \alpha_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j, j = 0, 1, ….., n are real numbers and for certain non-negative real numbers t_1, t_2 with $t_1 > t_2$ and $t_1 \ne 0$

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \ge 0,$$
$$b_r t_1 t_2 + b_{r-1}(t_1 - t_2) - b_{r-2} \ge 0, \quad r = 2,....,n$$

and for some $k \ge 1$
and

$$ka_n(t_1 - t_2) - a_{n-1} \ge 0$$
$$kb_n(t_1 - t_2) - b_{n-1} \ge 0$$
$$a_{-1} = a_{n+1} = 0 = b_{n+1} = b_{-1}$$

then all the zeros of $p(z)$ lie in

$$|z + (k-1)(t_1 - t_2)| \le R_1,$$

where

$$R_1 = \frac{1}{|\alpha_n|}\left[\left\{ka_n(t_1 - t_2) + a_n t_2 - \frac{a_1 t_2}{t_1^{n-1}} - \frac{a_0}{t_1^{n-1}}\right\} + \left\{kb_n(t_1 - t_2) + b_n t_2 - \frac{b_1 t_2}{t_1^{n-1}} - \frac{b_0}{t_1^{n-1}}\right\} + \left|a_1 t_1 t_2 + a_0(t_1 - t_2)\right|\frac{1}{t_1^n} + \left|b_1 t_1 t_2 + b_0(t_1 - t_2)\right|\frac{1}{t_1^n} + \left(|a_0| + |b_0|\right)\frac{t_2}{t_1^n}\right].$$

Remark 1.3. If we choose $t_2 = 0$ and $t_1 \equiv t$ in Theorem 1.6, we get the following result.

Corollary 1.2. Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $\alpha_j = a_j + ib_j$ where a_j and b_j, j = 0, 1, 2,….., n are real numbers and for certain non-negative real number t

$$a_{n-1}t^{n-1} \ge a_{n-2}t^{n-2} \ge .... \ge a_1 t > a_0$$
$$b_{n-1}t^{n-1} \ge b_{n-2}t^{n-2} \ge .... \ge b_1 t > b_0$$

and for some $k \ge 1$,

$$ka_n t^n \ge a_{n-1}t^{n-1}$$
$$kb_n t^n \ge b_{n-1}t^{n-1}.$$

Then all the zeros of $p(z)$ lie in

$$|z + (k-1)t| \le R_2,$$

where $$R_2 = \frac{1}{|\alpha_n|}\left[\begin{array}{l} k(a_n + b_n)t + \left(|a_0| + |b_0|\right)\frac{1}{t^{n-1}} \\ -(a_0 + b_0)\frac{1}{t^{n-1}} \end{array}\right].$$

Remark 1.4. If we consider all the coefficients in Corollary 1.2 to be real and positive, then we obtain Theorem 1.3 due to Aziz and Zargar [1].

Remark 1.5. Again, if we choose all the coefficients in Corollary 1 to be real and consider k=1 and t=1, we get Theorem B due to Joyal, Labelle and Rahman [5].

2. Proof of the Theorem

Proof of Theorem1.5. Consider the polynomial

$$\begin{aligned} F(z) &= (t_2 + z)(t_1 - z)p(z) \\ &= \left\{t_1 t_2 + (t_1 - t_2)z - z^2\right\}\left\{\begin{array}{l}\alpha_n z^n + \alpha_{n-1}z^{n-1} \\ +......+\alpha_1 z + \alpha_0\end{array}\right\} \\ &= -\alpha_n z^{n+2} + \left\{\alpha_n(t_1 - t_2) - \alpha_{n-1}\right\}z^{n+1} \\ &+ \left\{\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}\right\}z^n + ... \\ &+ \left\{\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0\right\}z^2 \\ &+ \left\{\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)\right\}z + \alpha_0 t_1 t_2 \end{aligned}$$

$$= -a_n z^{n+2} + \{a_n (t_1 - t_2) - a_{n-1}\} z^{n+1} + ...$$
$$+ \{a_1 t_1 t_2 + a_0 (t_1 - t_2)\} z + a_0 t_1 t_2 +$$
$$i\Big[-b_n z^{n+2} + \{b_n (t_1 - t_2) - b_{n-1}\} z^{n+1} + ...$$
$$+ \{b_1 t_1 t_2 + b_0 (t_1 - t_2)\} z + b_0 t_1 t_2\Big].$$

Further, let

$$G(z) = z^{n+2} F\left(\frac{1}{z}\right) = -\alpha_n + \{\alpha_n (t_1 - t_2) - \alpha_{n-1}\} z$$
$$+ \{\alpha_n t_1 t_2 + \alpha_{n-1} (t_1 - t_2) - \alpha_{n-2}\} z^2 + ...$$
$$+ \{\alpha_2 t_1 t_2 + \alpha_1 (t_1 - t_2) - \alpha_0\} z^n$$
$$+ \{\alpha_1 t_1 t_2 + \alpha_0 (t_1 - t_2)\} z^{n+1} + \alpha_0 t_1 t_2 z^{n+2}$$
$$= -a_n + \{a_n (t_1 - t_2) - a_{n-1}\} z$$
$$+ \{a_n t_1 t_2 + a_{n-1} (t_1 - t_2) - a_{n-2}\} z^2 + ...$$
$$+ \{a_2 t_1 t_2 + a_1 (t_1 - t_2) - a_0\} z^n$$
$$+ \{a_1 t_1 t_2 + a_0 (t_1 - t_2)\} z^{n+1} + a_0 t_1 t_2 z^{n+2}$$
$$+ i\Big[-b_n + \{b_n (t_1 - t_2) - b_{n-1}\} z$$
$$+ \{b_n t_1 t_2 + b_{n-1} (t_1 - t_2) - b_{n-2}\} z^2 + ...$$
$$+ \{b_2 t_1 t_2 + b_1 (t_1 - t_2) - b_0\} z^n$$
$$+ \{b_1 t_1 t_2 + b_0 (t_1 - t_2)\} z^{n+1} + b_0 t_1 t_2 z^{n+2}\Big] \qquad (3.1)$$
$$= -\alpha_n + \phi(z),$$

where

$$\phi(z) = \{a_n (t_1 - t_2) - a_{n-1}\} z$$
$$+ \{a_n t_1 t_2 + a_{n-1} (t_1 - t_2) - a_{n-2}\} z^2 + ...$$
$$+ \{a_2 t_1 t_2 + a_1 (t_1 - t_2) - a_0\} z^n$$
$$+ \{a_1 t_1 t_2 + a_0 (t_1 - t_2)\} z^{n+1}$$
$$+ a_0 t_1 t_2 z^{n+2} + i\Big[\{b_n (t_1 - t_2) - b_{n-1}\} z$$
$$+ \{b_n t_1 t_2 + b_{n-1} (t_1 - t_2) - b_{n-2}\} z^2 + ...$$
$$+ \{b_2 t_1 t_2 + b_1 (t_1 - t_2) - b_0\} z^n$$
$$+ \{b_1 t_1 t_2 + b_0 (t_1 - t_2)\} z^{n+1} + b_0 t_1 t_2 z^{n+2}\Big].$$

Now

$$|\phi(z)| \le |a_n (t_1 - t_2) - a_{n-1}||z|$$
$$+ |a_n t_1 t_2 + a_{n-1} (t_1 - t_2) - a_{n-2}||z|^2 + ...$$
$$+ |a_2 t_1 t_2 + a_1 (t_1 - t_2) - a_0||z|^n$$
$$+ |a_1 t_1 t_2 + a_0 (t_1 - t_2)||z|^{n+1}$$
$$+ |a_0 t_1 t_2||z|^{n+2} + |b_n (t_1 - t_2) - b_{n-1}||z|$$
$$+ |b_n t_1 t_2 + b_{n-1} (t_1 - t_2) - b_{n-2}||z|^2 + ...$$
$$+ |b_2 t_1 t_2 + b_1 (t_1 - t_2) - b_0||z|^n$$
$$+ |b_1 t_1 t_2 + b_0 (t_1 - t_2)||z|^{n+1} + |b_0 t_1 t_2||z|^{n+2}.$$

This gives after using hypothesis, for $|z| = 1/t_1$

$$|\phi(z)| \le (a_n + b_n) + (|a_0| - a_0)\frac{t_2}{t_1^{n+1}} + (|b_0| - b_0)\frac{t_2}{t_1^{n+1}}.$$

Clearly $\phi(z)$.

Therefore, by Schwarz's Lemma

$$|\phi(z)| \le (a_n + b_n) + (|a_0| - a_0)\frac{t_2}{t_1^{n+1}} + (|b_0| - b_0)\frac{t_2}{t_1^{n+1}}|z| t_1$$

for $|z| \le \frac{1}{t_1}$.

Hence from (3.1), if $M = \frac{t_2}{t_1^{n+1}}\left[(|a_0| - a_0) + (|b_0| - b_0)\right]$, then we get

$$|G(z)| \ge |\alpha_n| - |\phi(z)|$$
$$\ge |\alpha_n| - |\alpha_n + M||z| t_1 > 0,$$

If $|\alpha_n| \ge |\alpha_n + M||z| t_1$ for $|z| \le \frac{1}{t_1}$.

That is $|G(z)| > 0$, if $|z| \le \frac{|\alpha_n|}{|\alpha_n + M| t_1}$,

Thus in $|z| \le \frac{1}{t_1}$, $|G(z)| > 0$, if $|z| \le \frac{|\alpha_n|}{|\alpha_n + M| t_1}$.

Consequently, all the zeros of G(z) lie in $|z| \ge \frac{|\alpha_n|}{|\alpha_n + M| t_1}$.

As $F(z) = z^{n+2} G\left(\frac{1}{z}\right)$, we conclude that all the zeros of F(z) and hence all the zeros of p(z) lie in

$$|z| \le \frac{|\alpha_n + M|}{|\alpha_n|} t_1.$$

This completes the proof of the theorem 1.5.

Proof of Theorem 1.6. Consider the polynomial

$$f(z) = (t_2 + z)(t_1 - z)\, p(z)$$
$$= \left\{t_1 t_2 + (t_1 - t_2) z - z^2\right\}\left\{\begin{matrix}\alpha_n z^n + \alpha_{n-1} z^{n-1} \\ + ... + \alpha_1 z + \alpha_0\end{matrix}\right\}$$

$$=-\alpha_n z^{n+2}+\left\{\alpha_n(t_1-t_2)-\alpha_{n-1}\right\}z^{n+1}$$
$$+\left\{\alpha_n t_1 t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}\right\}z^n+......$$
$$+\left\{\alpha_2 t_1 t_2+\alpha_1(t_1-t_2)-a_0\right\}z^2+$$
$$\left\{\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)\right\}z+\alpha_0 t_1 t_2$$
$$=-\alpha_n z^{n+2}-\left\{\begin{matrix}k\alpha_n(t_1-t_2)-\alpha_n(t_1-t_2)\\-k\alpha_n(t_1-t_2)+\alpha_{n-1}\end{matrix}\right\}z^{n+1}$$
$$+\left\{\alpha_n t_1 t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}\right\}z^n+......$$
$$+\left\{\alpha_2 t_1 t_2+\alpha_1(t_1-t_2)-\alpha_0\right\}z^2$$
$$+\left\{\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)\right\}z+\alpha_0 t_1 t_2$$
$$=-\alpha_n z^{n+2}-\left\{\begin{matrix}(k-1)\alpha_n(t_1-t_2)\\-k\alpha_n(t_1-t_2)-\alpha_{n-1}\end{matrix}\right\}z^{n+1}$$
$$+\left\{\alpha_n t_1 t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}\right\}z^n+......$$
$$+\left\{\alpha_2 t_1 t_2+\alpha_1(t_1-t_2)-\alpha_0\right\}z^2$$
$$+\left\{\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)\right\}z+\alpha_0 t_1 t_2$$
$$=-a_n z^{n+2}-\left\{\begin{matrix}(k-1)a_n(t_1-t_2)\\-ka_n(t_1-t_2)+a_{n-1}\end{matrix}\right\}z^{n+1}$$
$$+\left\{a_n t_1 t_2+a_{n-1}(t_1-t_2)-a_0\right\}z^2$$
$$+\left\{a_1 t_1 t_2+a_0(t_1-t_2)\right\}z+a_0 t_1 t_2$$
$$+i\left[-b_n z^{n+2}-\left\{\begin{matrix}(k-1)b_n(t_1-t_2)\\-kb_n(t_1-t_2)+b_{n-1}\end{matrix}\right\}z^{n+1}\right.$$
$$+\left\{b_n t_1 t_2+b_{n-1}(t_1-t_2)-b_0\right\}z^2$$
$$\left.+\left\{b_1 t_1 t_2+b_0(t_1-t_2)\right\}z+b_0 t_1 t_2\right]$$

$$f(z)=(t_2+z)(t_1-z)\,p(z)$$
$$=\left\{t_1 t_2+(t_1-t_2)z-z^2\right\}\left\{\begin{matrix}\alpha_n z^n+\alpha_{n-1}z^{n-1}\\+......+\alpha_1 z+\alpha_0\end{matrix}\right\}$$
$$=-\alpha_n z^{n+2}+\left\{\alpha_n(t_1-t_2)-\alpha_{n-1}\right\}z^{n+1}$$
$$+\left\{\alpha_n t_1 t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}\right\}z^n+......$$
$$+\left\{\alpha_2 t_1 t_2+\alpha_1(t_1-t_2)-a_0\right\}z^2+$$
$$\left\{\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)\right\}z+\alpha_0 t_1 t_2$$
$$=-\alpha_n z^{n+2}-\left\{\begin{matrix}k\alpha_n(t_1-t_2)-\alpha_n(t_1-t_2)\\-k\alpha_n(t_1-t_2)+\alpha_{n-1}\end{matrix}\right\}z^{n+1}$$
$$+\left\{\alpha_n t_1 t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}\right\}z^n+......$$
$$+\left\{\alpha_2 t_1 t_2+\alpha_1(t_1-t_2)-\alpha_0\right\}z^2$$
$$+\left\{\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)\right\}z+\alpha_0 t_1 t_2$$
$$=-\alpha_n z^{n+2}-\left\{\begin{matrix}(k-1)\alpha_n(t_1-t_2)\\-k\alpha_n(t_1-t_2)-\alpha_{n-1}\end{matrix}\right\}z^{n+1}$$
$$+\left\{\alpha_n t_1 t_2+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}\right\}z^n+......$$
$$+\left\{\alpha_2 t_1 t_2+\alpha_1(t_1-t_2)-\alpha_0\right\}z^2$$
$$+\left\{\alpha_1 t_1 t_2+\alpha_0(t_1-t_2)\right\}z+\alpha_0 t_1 t_2$$
$$=-a_n z^{n+2}-\left\{\begin{matrix}(k-1)a_n(t_1-t_2)\\-ka_n(t_1-t_2)+a_{n-1}\end{matrix}\right\}z^{n+1}$$
$$+\left\{a_n t_1 t_2+a_{n-1}(t_1-t_2)-a_0\right\}z^2$$
$$+\left\{a_1 t_1 t_2+a_0(t_1-t_2)\right\}z$$
$$+a_0 t_1 t_2+i\left[-b_n z^{n+2}-\left\{\begin{matrix}(k-1)b_n(t_1-t_2)\\-kb_n(t_1-t_2)+b_{n-1}\end{matrix}\right\}z^{n+1}\right.$$
$$+\left\{b_n t_1 t_2+b_{n-1}(t_1-t_2)-b_0\right\}z^2$$
$$\left.+\left\{b_1 t_1 t_2+b_0(t_1-t_2)\right\}z+b_0 t_1 t_2\right]$$
$$=-a_n z^{n+2}-(k-1)a_n(t_1-t_2)z^{n+1}$$
$$+\left\{ka_n(t_1-t_2)-a_{n-1}\right\}z^{n+1}$$
$$+\left\{a_n t_1 t_2+a_{n-1}(t_1-t_2)-a_{n-2}\right\}z^n+......$$
$$+\left\{a_2 t_1 t_2+a_1(t_1-t_2)+a_0\right\}z^2$$
$$+\left\{a_1 t_1 t_2+a_0(t_1-t_2)\right\}z+a_0 t_1 t_2$$
$$+i\left[-b_n z^{n+2}-(k-1)b_n(t_1-t_2)z^{n+1}\right.$$
$$+\left\{b_n t_1 t_2+b_{n-1}(t_1-t_2)-b_{n-2}\right\}z^n+......$$
$$+\left\{b_2 t_1 t_2+b_1(t_1-t_2)-b_0\right\}z^2$$
$$\left.+\left\{b_1 t_1 t_2+b_0(t_1-t_2)\right\}z+b_0 t_1 t_2\right].$$

This gives

$$|f(z)|\geq|\alpha_n||z|^{n+1}|z+(k-1)(t_1-t_2)|-|\phi(z)|.$$

Therefore for $|z|>t_1$, we have

$$|f(z)|\geq|z|^{n+1}\left[|\alpha_n||z+(k-1)(t_1-t_2)|\right.$$
$$-|ka_n(t_1-t_2)-a_{n-1}|$$
$$-|a_n t_1 t_2+a_{n-1}(t_1-t_2)-a_{n-2}|\frac{1}{t_1}-......$$
$$-|a_2 t_1 t_2+a_1(t_1-t_2)-a_0|\frac{1}{t_1^{n-1}}$$
$$-|a_1 t_1 t_2+a_0(t_1-t_2)|\frac{1}{t_1^n}-|a_0 t_1 t_2|\frac{1}{t_1^{n+1}}$$
$$-|kb_n(t_1-t_2)-b_{n-1}|$$
$$-|b_n t_1 t_2+b_{n-1}(t_1-t_2)-b_{n-2}|\frac{1}{t_1}-......$$
$$-|b_2 t_1 t_2+b_1(t_1-t_2)-b_0|\frac{1}{t_1^{n-1}}$$
$$\left.-|b_1 t_1 t_2+b_0(t_1-t_2)|\frac{1}{t_1^n}-|b_0 t_1 t_2|\frac{1}{t_1^{n+1}}\right]$$
$$=|z|^{n+1}\left[|\alpha_n||z+(k-1)(t_1-t_2)|\right.$$
$$-\left\{\begin{matrix}ka_n(t_1-t_2)+a_n t_2\\-\frac{a_1 t_2}{t_1^{n-1}}-\frac{a_0}{t_1^{n-1}}\end{matrix}\right\}-\left\{\begin{matrix}kb_n(t_1-t_2)+b_n t_2\\-\frac{b_1 t_2}{t_1^{n-1}}-\frac{b_0}{t_1^{n-1}}\end{matrix}\right\}$$

$$-\left|a_1t_1t_2+a_0\left(t_1-t_2\right)\right|\frac{1}{t_1^n}$$

$$\left.-\left|b_1t_1t_2+b_0\left(t_1-t_2\right)\right|\frac{1}{t_1^n}-\left|a_0t_1t_2\right|\frac{1}{t_1^{n+1}}-\left|b_0t_1t_2\right|\frac{1}{t_1^{n+1}}\right]$$

$$>0,$$

if

$$\left|z+\left(k-1\right)\left(t_1-t_2\right)\right|$$

$$>\frac{1}{\left|\alpha_n\right|}\left[\left\{ka_n\left(t_1-t_2\right)+a_nt_2-\frac{a_1t_2}{t_1^{n-1}}-\frac{a_0}{t_1^{n-1}}\right\}\right.$$

$$+\left\{kb_n\left(t_1-t_2\right)+b_nt_2-\frac{b_1t_2}{t_1^{n-1}}-\frac{b_0}{t_1^{n-1}}\right\}$$

$$+\left|a_1t_1t_2+a_0\left(t_1-t_2\right)\right|\frac{1}{t_1^n}$$

$$\left.+\left|b_1t_1t_2+b_0\left(t_1-t_2\right)\right|\frac{1}{t_1^n}+\left(\left|a_0\right|+\left|b_0\right|\right)\frac{t_2}{t_1^n}\right]$$

$$=R_1.$$

Hence all the zeros of $f\left(z\right)$ whose modulus is greater than t_1 lie in the circle

$$\left|z+\left(k-1\right)\left(t_1-t_2\right)\right|\le R_1.$$

Since all the zeros whose modulus is less than t_1 already lies in this circle, we conclude that all the zeros of $f\left(z\right)$ and hence $p\left(z\right)$ lie in

$$\left|z+\left(k-1\right)\left(t_1-t_2\right)\right|\le R_1.$$

This completes the proof of the Theorem1.6.

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