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On the Eneström-Kakeya Theorem

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Abstract In this paper, we prove some extensions of the Eneström-Kakeya theorem by relaxing the hypothesis in different ways which in turn generalizes a result of Aziz and Zargar [*Some extensions of Eneström-Kakeya Theorem*, Glasnik Matematički, 31(1996), 239-244].

Keywords: *polynomial, zeros, Eneström-Kakeya theorem*

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1. Introduction and Statement of Results

Finding the roots of a polynomial is a long standing classical problem [3,7]. The various results in the analytic theory of polynomials concerning the number of zeros in a region have been frequently investigated. It is an interesting area of research for engineers as well as mathematicians and many results on the same topic are available in the literature. Over last five decades, a large number of research papers, e.g, [1,4,5,6,8,9,13,14] and monographs [10,12,15] have been published. Polynomials in various forms have recently come under extensive revision because of their applications in linear control systems, signal processing, electrical networks, coding theory and several areas of physical sciences, where among others location of zeros and stability problems arise in a natural way. Existing results in the literature also show that there is need to find bounds for special polynomials, for example, those having restrictions on the coefficient and there is always need for refinement of results in this subject.

The following result is well known in the theory of the distribution of zeros of polynomials.

Theorem A. *If* $P(z):=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n such that*

$$a_n \ge a_{n-1} \ge a_{n-2} \ge \ldots \ge a_1 \ge a_0 > 0,$$

then $P(z)$ does not vanish in $|z|>1$.

If we apply this result to the polynomial $P(kz), k>0$, then it can be restated as:

Theorem B. *Let* $P(z):=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that*

$$a_n k^n \ge a_{n-1} k^{n-1} \ge a_{n-2} k^{n-2} \ge \ldots \ge a_1 k \ge a_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \le k. \tag{1}$$

The Eneström-Kakeya theorem is a very strong tool to find the region in the complex plane containing all the zeros of a class of polynomials. It has been used to analyze overflow oscillation of discrete-time dynamical system [11], to investigate the properties of orthogonal wavelets [9], to determine the asymptotic behavior of zeros of the Daubechies filter [16], in addition for application to a model of high energy collisions [2].

In the literature, [1,4,5,6,14], there exist extensions and generalizations of Eneström- Kakeya theorem.

Joyal et al. [8] obtained the following generalization, by considering the coefficients to be real, instead of being only positive.

Theorem C. *Let* $P(z):=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that*

$$a_n \ge a_{n-1} \ge a_{n-2} \ge \ldots \ge a_1 \ge a_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \le \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}.$$

Aziz and Zargar [1] also relaxed the hypothesis of Eneström-Kakeya theorem in a different way and proved the following results.

Theorem D. *Let* $P(z):=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for some* $k \ge 1$,

$$ka_n \ge a_{n-1} \ge a_{n-2} \ge \ldots \ge a_1 \ge a_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$|z+k-1| \le k.$$

Theorem E. *Let* $P(z):=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for some* $k \ge 1$,

$$ka_n \geq a_{n-1} \geq a_{n-2} \geq \ldots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z+k-1| \leq \frac{ka_n - a_0 + |a_0|}{|a_n|}.$$

In this paper, we prove more general result by relaxing the hypothesis in different ways which includes Theorem E as a special case.

Theorem 1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that* $\alpha_j = a_j + ib_j$ *where* a_j *and* b_j, $j = 0,1,\ldots,n$ *are real numbers. If for some positive integers* $\lambda, \mu \leq n$, *and for some real numbers* $0 < \rho_1 \leq 1, 0 < \rho_2 \leq 1, k \geq 1,$

$$k^{n-\lambda+1}a_n \geq k^{n-\lambda}a_{n-1} \geq k^{n-\lambda-1}a_{n-2} \ldots \geq k^2 a_{\lambda+1} \geq ka_\lambda \geq a_{\lambda-1} \geq \ldots a_1 \geq \rho_1 a_0,$$

$$k^{n-\mu+1}a_n \geq k^{n-\mu}a_{n-1} \geq k^{n-\mu-1}a_{n-2} \ldots \geq k^2 a_{\mu+1} \geq ka_\mu \geq a_{\mu-1} \geq \ldots a_1 \geq \rho_2 b_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z+k-1| \leq \frac{1}{|a_n|}\left\{(a_n+b_n) - (\rho_1 a_0 + \rho_2 b_0) + (1-\rho_1)|a_0| + (1-\rho_2)|b_0| + (|a_0|+|b_0|) + (k-1)\begin{pmatrix} \sum_{j=\lambda}^{n}(a_j + |a_j|) \\ + \sum_{j=\mu}^{n}(b_j+|b_j|) - |a_n| - |b_n| \end{pmatrix}\right\}. \quad (2)$$

Remark 1. Theorem E is a special case of Theorem 1, if we take all coefficients of $P(z)$ are real, $\rho_1 = 1$ and $\lambda = n$.

The following Corollary immediately follows from Theorem 1.

Corollary 1. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that* $\alpha_j = a_j + ib_j$ *where* a_j *and* b_j, $j = 0,1,\ldots,n$ *are real numbers. If for some positive integers* $\lambda, \mu \leq n$, *and for some real numbers* $0 < \rho_1 \leq 1, 0 < \rho_2 \leq 1, k \geq 1,$

$$a_n \geq a_{n-1} \geq a_{n-2} \ldots \geq a_{\lambda+1} \geq ka_\lambda \geq a_{\lambda-1} \geq \ldots \geq a_1 \geq \rho_1 a_0,$$

$$b_n \geq b_{n-1} \geq b_{n-2} \ldots \geq b_{\lambda+1} \geq kb_\mu \geq b_{\mu-1} \geq \ldots \geq a_1 \geq \rho_2 b_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z+k-1| \leq \frac{1}{|a_n|}\left\{(a_n+b_n) - (\rho_1 a_0 + \rho_2 b_0) + (1-\rho_1)|a_0| + (1-\rho_2)|b_0| + (|a_0|+|b_0|) + (k-1)\begin{pmatrix} \sum_{j=\lambda}^{n}(a_j + |a_j|) \\ + \sum_{j=\mu}^{n}(b_j+|b_j|) - |a_n| - |b_n| \end{pmatrix}\right\}.$$

If we choose $\lambda = \mu = n-1$ and $a_0, b_0 > 0$ in Theorem 1, we have the following:

Corollary 2. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that* $\alpha_j = a_j + ib_j$ *where* a_j *and* b_j, $j = 0,1,\ldots,n$ *are real numbers. If for some real numbers* $0 < \rho_1 \leq 1, 0 < \rho_2 \leq 1, k \geq 1,$

$$k^2 a_n \geq ka_{n-1} \geq a_{n-2} \geq a_{n-3} \geq \ldots \geq a_1 \geq \rho_1 a_0 > 0,$$

$$k^2 b_n \geq kb_{n-1} \geq b_{n-2} \geq b_{n-3} \geq \ldots \geq b_1 \geq \rho_2 b_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$|z+k-1| \leq \frac{1}{|a_n|}\begin{Bmatrix} 2((1-\rho_1)a_0 + (1-\rho_2)b_0) \\ +k(a_n+b_n) + 2(k-1)(a_{n-1}+b_{n-1}) \end{Bmatrix}.$$

Also, if we put $\lambda = \mu = \rho_1 = \rho_2 = 1$ and $a_0, b_0 > 0$ in Theorem 1, we get

Corollary 3. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that* $\alpha_j = a_j + ib_j$ *where* a_j *and* b_j, $j = 0,1,\ldots,n$ *are real numbers. If for some real number* $k \geq 1,$

$$k^n a_n \geq k^{n-1}a_{n-1} \geq \ldots \geq ka_1 \geq a_0 > 0,$$

$$k^n b_n \geq k^{n-1}b_{n-1} \geq \ldots \geq kb_1 \geq b_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$|z+k-1| \leq \frac{1}{|a_n|}\left\{k(a_n+b_n) + 2(k+1)\sum_{j=1}^{n-1} a_j\right\} \quad (3)$$

On combining Theorem B and by taking all coefficients of $P(z)$ to be real in the Corollary 2, the following is immediate:

Corollary 4. *Let* $P(z) := \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n, if for some* $k \geq 1,$

$$k^n a_n \geq k^{n-1}a_{n-1} \geq \ldots \geq ka_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ *lie in the intersection of the discs represented by (1) and (3).*

2. Proof of the Theorem 1

Consider the polynomial

$$\begin{aligned} H(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots \\ &\quad + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \ldots \\ &\quad + (a_1 - a_0)z + a_0 \\ &\quad + i\begin{Bmatrix} -b_n z^{n+1} + (b_n - b_{n-1})z^n + \ldots \\ +(b_{\mu+1} - b_\mu)z^{\mu+1} + (b_\mu - b_{\mu-1})z^\mu + \ldots \\ +(b_1 - b_0)z + b_0 \end{Bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= -a_n z^n (z+k-1) + (ka_n - a_{n-1}) z^n \\
&\quad + (ka_{n-1} - a_{n-2} - (k-1)a_{n-1}) z^{n-1} \\
&\quad + (ka_{n-2} - a_{n-3} - (k-1)a_{n-2}) z^{n-2} + \ldots \\
&\quad + (ka_{\lambda+1} - a_{\lambda} - (k-1)a_{\lambda+1}) z^{\lambda+1} \\
&\quad + (ka_{\lambda} - a_{\lambda-1} - (k-1)a_{n-1}) z^{\lambda} + \ldots \\
&\quad + [(a_1 - \rho_1 a_0) + (\rho_1 a_0 - a_0)] z + a_0 \\
&\quad + i \left\{ \begin{aligned}
&-b_n z^n (z+k-1) + (kb_n - b_{n-1}) z^n \\
&+ (kb_{n-1} - b_{n-2} - (k-1)b_{n-1}) z^{n-1} \\
&+ (kb_{n-2} - b_{n-3} - (k-1)b_{n-2}) z^{n-2} + \ldots \\
&+ (kb_{\mu+1} - b_{\mu} - (k-1)b_{\mu+1}) z^{\mu+1} \\
&+ (kb_{\mu} - b_{\mu-1} - (k-1)b_{\mu}) z^{\mu} + \ldots \\
&+ [(b_1 - \rho_2 b_0) + (\rho_2 b_0 - b_0)] z + b_0
\end{aligned} \right\}
\end{aligned}
$$

$$
= z^n \left[\begin{aligned}
&a_n (z+k-1) + (ka_n - a_{n-1}) + (ka_{n-1} - a_{n-2}) \frac{1}{z} \\
&+ (ka_{n-2} - a_{n-3}) \frac{1}{z^2} + \ldots + (ka_{\lambda+1} - a_{\lambda}) \frac{1}{z^{n-\lambda-1}} \\
&+ (ka_{\lambda} - a_{\lambda-1}) \frac{1}{z^{n-\lambda}} + \ldots + (a_1 - \rho_1 a_0) \frac{1}{z^{n-1}} \\
&+ (\rho_1 - 1) a_0 \frac{1}{z^{n-1}} + a_0 \frac{1}{z^n} - (k-1) \left\{ \begin{aligned} &\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots \\ &+ \frac{a_{\lambda+1}}{z^{n-\lambda-1}} + \frac{a_{\lambda}}{z^{n-\lambda}} \end{aligned} \right\} \\
&+ i \left\{ \begin{aligned}
&b_n (z+k+1) + (kb_n - b_{n-1}) + (kb_{n-1} - b_{n-2}) \frac{1}{z} \\
&+ (kb_{n-2} - b_{n-3}) \frac{1}{z^2} + \ldots + (ka_{\mu+1} - b_{\mu}) \frac{1}{z^{n-\mu-1}} \\
&+ (kb_{\mu} - b_{\mu-1}) \frac{1}{z^{n-\mu}} + \ldots + (b_1 - \rho_2 b_0) \frac{1}{z^{n-1}} \\
&+ (\rho_2 - 1) b_0 \frac{1}{z^{n-1}} + b_0 \frac{1}{z^n}
\end{aligned} \right\} \\
&- (k-1) \left\{ \frac{b_{-1}}{z} + \frac{b_{n-2}}{z^2} + \ldots + \frac{b_{\mu+1}}{z^{n-\mu-1}} + \frac{b_{\mu}}{z^{n-\mu}} \right\}
\end{aligned} \right]
$$

Let $|z| > 1,$ so that $\frac{1}{|z|^{n-j}} < 1, 0 \le j \le n$ and we have

$$
\begin{aligned}
|H(z)| &\ge |z|^n \Big[|\alpha_n| |z+k-1| \\
&\quad - \Big\{ [ka_n - a_{n-1}) + \frac{|ka_{n-1} - a_{n-2}|}{|z|} \\
&\quad + \frac{|ka_{n-2} - a_{n-3}|}{|z|^2} + \cdots + \frac{|ka_{\lambda+1} - a_{\lambda}|}{|z|^{n-\lambda-1}} \\
&\quad + \frac{|ka_{\lambda} - a_{\lambda-1}|}{|z|^{n-\lambda}} + \ldots + \frac{|a_1 - \rho_1 a_0|}{|z|^{n-1}} + \frac{|\rho_1 - 1||a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \\
&\quad + |k-1| \left(\frac{|a_{n-1}|}{|z|} + \frac{|a_{n-2}|}{|z|^2} + \ldots + \frac{|a_{\lambda+1}|}{|z|^{n-\lambda-1}} + \frac{|a_{\lambda}|}{|z|^{n-\lambda}} \right) \\
&\quad + |kb_n - b_{-1}| + \frac{|kb_{n-1} - b_{n-2}|}{|z|} \\
&\quad + \frac{|kb_{n-2} - b_{n-3}|}{|z|^2} + \ldots + \frac{|kb_{\mu+1} - b_{\mu}|}{|z|^{n-\mu-1}} \\
&\quad + \frac{|kb_{\mu} - b_{\mu-1}|}{|z|^{n-\mu}} + \ldots + \frac{|b_1 - \rho_2 b_0|}{|z|^{n-1}} + \frac{|\rho_2 - 1||b_0|}{|z|^{n-1}} + \frac{|b_0|}{|z|^n} \\
&\quad + |k-1| \left(\frac{|b_{n-1}|}{|z|} + \frac{|b_{n-2}|}{|z|^2} + \ldots + \frac{|b_{\mu+1}|}{|z|^{n-\mu-1}} + \frac{|b_{\mu}|}{|z|^{n-\mu}} \right) \Big\} \Big] \\
&> |z|^n \Big[|\alpha_n| |z+k-1| - \Big\{ [ka_n - a_{n-1}) \\
&\quad + |ka_{n-1} - a_{n-2}| + |ka_{n-2} - a_{n-3}| + \ldots \\
&\quad + |ka_{\lambda+1} - a_{\lambda}| + |ka_{\lambda} - a_{\lambda-1}| + \ldots \\
&\quad + |a_1 - \rho_1 a_0| + |\rho_1 - 1||a_0| + |a_0| \\
&\quad + |k-1| \left(|a_{n-1}| + |a_{n-2}| + \ldots + |a_{\lambda+1}| + |a_{\lambda}| \right) \\
&\quad + |kb_n - b_{-1}| + |kb_{n-1} - b_{n-2}| + |kb_{n-2} - b_{n-3}| \\
&\quad + \ldots + |kb_{\mu+1} - b_{\mu}| + |kb_{\mu} - b_{\mu-1}| + \ldots \\
&\quad + |b_1 - \rho_2 b_0| + |\rho_2 - 1||b_0| + |b_0| \\
&\quad + |k-1| \left(|b_{n-1}| + |b_{n-2}| + \ldots + |b_{\mu+1}| + |b_{\mu}| \right) \Big\} \Big] \\
&= |z|^n \Big[|\alpha_n| |z+k-1| - \Big\{ a_n - \rho_1 a_0 + (1-\rho_1)|a_0| \\
&\quad + |a_0| + (k-1) \left(\textstyle\sum_{j=\mu}^{n} \left(a_j + |a_j| \right) - |a_n| \right) \\
&\quad + b_n - \rho_2 b_0 + (1-\rho_2)|b_0| + |b_0| \\
&\quad + (k-1) \left(\textstyle\sum_{j=\mu}^{n} \left(b_j + |b_j| \right) - |b_n| \right) \Big\} \Big] \\
&= |z|^n \Big[|\alpha_n| |z+k-1| - \Big\{ (a_n + b_n) - (\rho_1 a_0 + \rho_2 b_0) \\
&\quad + (1-\rho_1)|a_0| + (1|-\rho_2)|b_0| + (|a_0| + |b_0|) \\
&\quad + (k-1) \left(\textstyle\sum_{j=\mu}^{n} \left(a_j + |a_j| \right) \right. \\
&\quad \left. + \textstyle\sum_{j=\mu}^{n} \left(b_j + |b_j| \right) - |a_n| - |b_n| \right) \Big\} \Big] \\
&> 0,
\end{aligned}
$$

if

$$
\begin{aligned}
|z+k-1| > \frac{1}{|a_n|} \Big\{ &(a_n + b_n) - (\rho_1 a_0 + \rho_2 b_0) \\
&+ (1-\rho_1)|a_0| + (1-\rho_2)|b_0| \\
&+ (|a_0| + |b_0|) + (k-1) \left(\textstyle\sum_{j=\mu}^{n} \left(a_j + |a_j| \right) \right. \\
&\left. + \textstyle\sum_{j=\mu}^{n} \left(b_j + |b_j| \right) - |a_n| - |b_n| \right\}
\end{aligned}
$$

This shows that if $|z| > 1,$ then $|H(z)| > 0$ if

$$
\begin{aligned}
|z+k-1| > \frac{1}{|a_n|} \Big\{ &(a_n + b_n) - (\rho_1 a_0 + \rho_2 b_0) \\
&+ (1-\rho_1)|a_0| + (1-\rho_2)|b_0| \\
&+ (|a_0| + |b_0|) + (k-1) \left(\textstyle\sum_{j=\mu}^{n} \left(a_j + |a_j| \right) \right. \\
&\left. + \textstyle\sum_{j=\mu}^{n} \left(b_j + |b_j| \right) - |a_n| - |b_n| \right\}
\end{aligned}
$$

Hence all the zeros of $H(z)$ with $|z|>1$ lie

$$\begin{aligned}|z+k-1| \le \frac{1}{|a_n|}\Big\{&(a_n+b_n)-(\rho_1 a_0+\rho_2 b_0)\\ &+(1-\rho_1)|a_0|+(1-\rho_2)|b_0|\\ &+(|a_0|+|b_0|)+(k-1)\left(\sum_{j=\mu}^{n}\left(a_j+|a_j|\right)\right.\\ &\left.+\sum_{j=\mu}^{n}\left(b_j+|b_j|\right)-|a_n|-|b_n|\right\}\end{aligned} \qquad (1)$$

But those zeros of $H(z)$ whose modulus is less than or equal to 1 already satisfy the inequality (1). Since all the zeros of $P(z)$ are also the zeros of $H(z)$, therefore it follows that all the zeros of $P(z)$ lie in the circle defined by (1) and this completes the proof of Theorem 1.

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