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Common and Coincidence Fixed Point Theorems for Asymptotically Regular Mappings in Hilbert Spaces

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Abstract In this paper we prove common and coincidences fixed point theorems for asymptotically regular mappings under various contractive conditions on a Hilbert space setting. We also study the well – posedness of a common fixed point problem. Our results generalize several well known results in the literature.

Keywords: *asymptotically regular mappings, common and coincidences fixed points, weakly compatible mappings, Hilbert spaces*

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1. Introductions and Preliminaries

Most of fixed point theorems for mappings in metric spaces satisfying different contraction conditions may be extended to the abstract spaces like Hilbert spaces, Banach spaces and locally convex spaces etc., with some modifications Banach fixed point theorem and its applications are well known. Many authors have extended this theorem, introducing more general contractive conditions, which imply the existence of a fixed point. Almost all of conditions imply the asymptotic regularity of the mappings under considerations. So the investigation of the asymptotically regular maps play an important role in fixed point theory.

Sharma and Yuel [14] and Guay and Singh [7] were among the first who used the concept of asymptotic regularity to prove fixed point theorems for wider class of mappings than a class of mappings introduced and studied by Ćirić [6].

The purpose of this paper is to prove some common and coincidences fixed point theorems in Hilbert spaces and we study the well- posedness of their fixed point problem.

Definition 1.1. [1]. A self mapping T on a closed subset of a Hilbert space H is said to be asymptotically regular at a point x in H, if

$$\left\| T^n x - T^{n+1} x \right\| \to 0 \;\; as \;\; n \to \infty,$$

where $T^n x$ denotes the nth iterate of T at x.

Definition 1.2. [1]. Let C be a closed subset of a Hilbert space H . A sequence $\{x_n\}$ in C is said to be asymptotically T - regular if $\|x_n - Tx_n\| \to 0 \;\; as \;\; n \to \infty.$

Definition 1.3. [5]. A pair of mappings (f, T) on a Hilbert space H is said to be weakly compatible if f and T commute at their coincidence point (i.e. f T x = T f x whenever f x = T x). A point $y \in H$ is called point of coincidence of two self –mappings f and T on H if there exists a point $x \in H$ such that $y = Tx = fx.$

The following lemma was given in [5] in a metric space setting.

Lemma 1.1. Let X be a non-empty set and the mappings $T, f: X \to X$ have a unique point of coincidence v in X. If the pair (f, T) is weakly compatible then T and f have a unique common fixed point.

Let H be a Hilbert space,T and f be self –mappings on H with $T(H) \subset f(H)$ and $x_0 \in H$. Choose a point $x_1 \in H$ such that $fx_1 = Tx_0$. This can be done since $T(H) \subset f(H)$ .Continuing this process, having chosen $x_1, \ldots\ldots, x_n$, we choose x_{n+1} in H such that

$$fx_{n+1} = Tx_n, \quad n = 0,1,2,\ldots.$$

The sequence $\{fx_n\}$ is called a T –sequence with initial point x_0 .

Definition1.4. Let T and f be self – mappings on a Hilbert space H , with $T(H) \subset f(H)$ and $x_0 \in H$. A mapping T is said to be asymptotically f – regular at a point x_0 if :

$$\left\| f x_n - f x_{n+1} \right\| \to 0 \quad as \quad n \to \infty,$$

Where $\{f x_n\}$ is a T – sequence with initial point x_0.

We know that a Banach space is a Hilbert space iff its norm satisfies the parallelogram law i.e., every $x, y \in H$,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2), \tag{1.1}$$

which implies,

$$\|x+y\|^2 \le 2\,(\|x\|^2 + \|y\|^2) \qquad (1.2)$$

2. Common Fixed Point Theorems

S. T. Patel and at.al. [9] gave the following theorem:

Theorem 2.1. Let C be a closed subset of a Hilbert space H and T be a mapping on H into it self satisfying

$$\begin{aligned}\|Tx-Ty\|^2 &\le \alpha\,\|x-Tx\|^2 + \beta\,\|y-Ty\|^2 \\ &+\gamma\,\|x-y\|^2 + \delta \min\left\{\|x-Ty\|^2, \|y-Tx\|^2\right\} \qquad (2.1)\\ &+\eta \frac{\|y-Tx\|^2}{1+\|x-Tx\|\|x-Ty\|},\end{aligned}$$

for all x, y in H, where α,β,γ *and* δ are non –negative reals with $\alpha+\beta+\gamma+4\eta<1$. Then T has a unique fixed point in H.

The purpose of this section is to extend Theorem 2.1 to the case of two mappings in a Hilbert space as the following.

Theorem 2.2. Let C be a closed subset of a Hilbert space H and S,T be mappings on C into itself satisfying:

$$\begin{aligned}\|Sx-Ty\|^2 &\le \alpha\,\|x-Sx\|^2 + \beta\,\|y-Ty\|^2 \\ &+\gamma\,\|x-y\|^2 + \delta \min\left\{\|x-Ty\|^2, \|y-Sx\|^2\right\} \qquad (2.2)\\ &+\eta \frac{\|y-Sx\|^2}{1+\|x-Sx\|\|x-Ty\|},\end{aligned}$$

for all $x, y \in C$, *where* α,β,γ *and* δ are non – negative reals with $\alpha+\beta+\gamma+4\eta<1$. Then S and T have a unique common fixed point in C.

Proof. Let $x_0 \in C$, we define a sequence $\{x_n\}$ as follows

$$x_{2n+1} = Sx_{2n}\ ,\ x_{2n+2} = Tx_{2n+1}\ ,\ n=0,1,2,....$$

From (2.2), we have

$$\begin{aligned}\|x_{2n+1}-x_{2n}\|^2 &= \|Sx_{2n}-Tx_{2n-1}\|^2 \\ &\le \alpha\|x_{2n}-Sx_{2n}\|^2 + \beta\|x_{2n-1}-Tx_{2n-1}\|^2 \\ &+\gamma\|x_{2n}-x_{2n-1}\|^2 \\ &+\delta\min\left\{\|x_{2n}Tx_{2n-1}\|^2, \|x_{2n-1}-Sx_{2n}\|^2\right\} \\ &+\eta\frac{\|x_{2n-1}-Sx_{2n}\|^2}{1+\|x_{2n}-Sx_{2n}\|\|x_{2n}-Tx_{2n-1}\|} \\ &= \alpha\|x_{2n}-x_{2n+1}\|^2 + \beta\|x_{2n-1}-x_{2n}\|^2 + \gamma\|x_{2n}-x_{2n-1}\|^2 \\ &+\delta\min\left\{\|x_{2n}-x_{2n}\|^2, \|x_{2n-1}-x_{2n+1}\|^2\right\} \\ &+\eta\frac{\|x_{2n-1}-x_{2n+1}\|^2}{1+\|x_{2n}-x_{2n+1}\|\|x_{2n}-x_{2n}\|} \\ &\le \alpha\|x_{2n}-x_{2n+1}\|^2 + \beta\|x_{2n-1}-x_{2n}\|^2 + \gamma\|x_{2n}-x_{2n-1}\|^2 \\ &+\eta\|x_{2n-1}-x_{2n+1}\|^2\end{aligned}$$

Thus we have

$$\|x_{2n}-x_{2n+1}\|^2 \le \frac{\beta+\gamma+2\eta}{1-\alpha-2\eta}\|x_{2n-1}-x_{2n}\|^2.$$

Putting $k=\dfrac{\beta+\gamma+2\eta}{1-\alpha-2\eta}<1$

Then, we have

$$\|x_{2n}-x_{2n+1}\|^2 \le k\|x_{2n-1}-x_{2n}\|^2.$$

Proceeding in this way, we get

$$\|x_{2n}-x_{2n+1}\|^2 \le k^n\|x_0-x_1\|^2\ ,\ n=1,2,....$$

For any positive integer p, one gets :

$$\begin{aligned}\|x_n-x_{n+p}\| &\le \|x_n-x_{n+1}\| + \|x_{n+1}-x_{n+2}\| \\ &+ ....+\|x_{n+p-1}-x_{n+p}\| \\ &\le (k^n+k^{n+1}+........+k^{n+p-1})\|x_0-x_1\| \\ &\le \frac{k^n}{1-k}\|x_0-x_1\| \quad \to 0.\end{aligned}$$

Thus $\|x_n-x_{n+p}\| \ \to 0$ *as* $n \to \infty$.

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is closed subset of H, then there exists an element $v\in C$ such that

$$\lim_{n\to\infty} x_n = v.$$

Now further, we have

$$\begin{aligned}\|v-Tv\|^2 &= \|(v-x_{2n+1})+(x_{2n+1}-Tv)\|^2 \\ &\le \|v-x_{2n+1}\|^2 + \|x_{2n+1}-Tv\|^2 \\ &\quad + 2\,\mathrm{Re} < v-x_{2n+1}, x_{2n+1}-Tv > \\ &\le \|v-x_{2n+1}\|^2 + \|Sx_{2n}-Tv\|^2 \\ &\quad + 2\,\mathrm{Re} < v-x_{2n+1}, x_{2n+1}-Tv > \\ &\le \|v-x_{2n+1}\|^2 + \alpha\|x_{2n}-x_{2n+1}\|^2 \\ &\quad + \beta\|v-Tv\|^2 + \gamma\|x_{2n}-v\|^2 \\ &\quad + \delta\min\left\{\|x_{2n}-Tv\|^2, \|v-x_{2n+1}\|^2\right\} \\ &\quad + \eta\frac{\|v-x_{2n+1}\|^2}{1+\|x_{2n}-x_{2n+1}\|\|x_{2n}-Tv\|} \\ &\quad + 2\,\mathrm{Re} < v-x_{2n+1}, x_{2n+1}-Tv >.\end{aligned}$$

As $n\to\infty$, $x_{2n}\to v$, $x_{2n+1}\to v$,

We have

$$2\,\mathrm{Re} < v-x_{2n+1},\ x_{2n+1}-Tv > \to 0.$$

Then

$$\|v-Tv\|^2 \le \beta\,\|v-Tv\|^2.$$

This implies that v =T v, since $\beta<1$.

Similarly we get v = S v. Then v is a common fixed point of S and T.

For the uniqueness, let $u \in C$ be another fixed point of S and T, where $u \neq v$, then

$$\|u-v\|^2 = \|Sv - Tu\|^2$$
$$= \alpha \| v - Sv\|^2 + \beta \| u - Tu\|^2 + \gamma \| u - v\|^2$$
$$+\delta \min\left\{\|v-Tu\|^2, \|u-Sv\|^2\right\} + \eta \frac{\|u-Sv\|^2}{1+\|v-Sv\|\|v-Tu\|}$$
$$\leq (\gamma+\delta+\eta)\|u-v\|^2,$$

Since $\gamma+\delta+\eta<1$, so u = v i.e., the common fixed point is unique.

Next, we extend Theorem 2.2 to the case of pair S^p and T^q, where p and q are some positive integers and to the case of a sequence of mappings satisfying the same contractive condition (2.2).

Theorem 2.3. Let C be a closed subset of a Hilbert space H and S,T be mappings on C into itself satisfying

$$\left\|S^p x - T^q y\right\|^2 \leq \alpha \left\|x - S^p x\right\|^2 + \beta \left\|y - T^q y\right\|^2$$
$$+\gamma \|x-y\|^2 + \delta \min\left\{\left\|x-T^q y\right\|^2, \left\|y-S^p x\right\|^2\right\} \quad (2.3)$$
$$+\eta \frac{\left\|y-S^p x\right\|^2}{1+\left\|x-S^p x\right\|\left\|x-T^q y\right\|},$$

for all $x, y \in C,\ p,\ q \in (0,1)$, *where* $\alpha,\beta,\gamma,\delta$ *and* η are nonnegative reals with $\alpha+\delta+\eta<1$. Then S and T have a unique common fixed point.

Proof. Since S^p *and* T^q satisfies all the conditions of theorem 2.2. Hence S^p *and* T^q have a unique common fixed point, we assume that they have a common fixed point v

$$S^p v = v \Rightarrow S(S^p v) = Sv,$$
$$S^p(Sv) = Sv.$$
$$\text{if } Sv = x_0, \text{then } S^p(x_0) = x_0,$$

So S v is a fixed point of S^p. Similarly we can show that T v is a fixed point of T^q.i.e., $T^q(Tv) = Tv$. Now we have

$$\| v - Tv\|^2 = \left\|S^p v - T^q(Tv)\right\|^2$$
$$\leq \alpha\left\|v - S^p v\right\|^2 + \beta\left\|Tv - T^q(Tv)\right\|^2 + \gamma\|v-Tv\|^2$$
$$+\delta \min\left\{\left\|v-T^q(Tv)\right\|^2, \left\|Tv - S^p v\right\|^2\right\}$$
$$+\eta \frac{\left\|Tv-S^p v\right\|^2}{1+\left\|v-S^p v\right\| \left\|v-T^q(Tv)\right\|}$$
$$\leq \gamma\|v-Tv\|^2 + \delta\min\left\{\|v-Tv\|^2, \|Tv-v\|^2\right\}$$
$$+\eta\|Tv-v\|^2.$$

Then we have:

$$\| v - Tv\|^2 \leq (\gamma+\delta+\eta)\|Tv-v\|^2 \ , \ \gamma+\delta+\eta<1,$$

So, we have v = Tv.

On the same way we can prove that S v =v. So v is a common fixed point of S and T.

To prove the uniqueness, let $v \neq w$ be another common fixed point of S and T. Then clearly w is also a common fixed point of S^p *and* T^q. So from Theorem 2.3 S^p *and* T^q have a common fixed point. Therefore w= v. Hence S and T have a unique common fixed point.

Hence we have proved that if x_0 is unique common fixed point of S^p *and* T^q for all p, q > 0, then x_0 is unique common fixed point of S and T.

Theorem 2.4. Let C be a closed subset of a Hilbert space H and let $\{F_i\}$ be a sequence of mapping on C converging point wise to F satisfying

$$\|F_i x - F_i y\|^2 \leq \alpha\|x-F_i x\|^2 + \beta\|y-F_i y\|^2 + \gamma\|x-y\|^2$$
$$+\delta\min\left\{\|x-F_i y\|^2, \|y-F_i y\|^2\right\}$$
$$+\eta\frac{\|y-F_i x\|^2}{1+\|x-F_i x\| \ \|x-F_i y\|},$$

for all x, y in C, where $\alpha,\beta,\gamma,\delta$ *and* η are nonnegative reals in [0,1] with $\alpha+\beta+\gamma+4\eta<1$. If F_i has a fixed point v_i and F has a fixed point v. Then the sequence $\{v_n\}$ converges to v.

Proof. Since it is given $F_i v_i = v_i$ *and* $Fv = v$.

Now

$$\| v - v_n\|^2 = \| Fv - F_n v_n \|^2$$
$$= \| (Fv - F_n v) + (F_n v - F_n v_n) \|^2$$
$$\leq \| Fv - F_n v \|^2 + \|F_n v - F_n v_n\|^2$$
$$+2\,\mathrm{Re} < Fv - F_n v, F_n v - F_n v_n >$$
$$\leq \|Fv-F_n v\|^2 + \alpha\|v-F_n\|^2 + \beta\|v_n - F_n v_n\|^2$$
$$+\gamma\|v-v_n\|^2 + \delta\min\left\{\|v-F_n v\|^2, \|v_n F_n v\|^2\right\}$$
$$+\eta\frac{\|v_n - F_n v\|^2}{1+\|v-F_n v\|\|v-F_n v_n\|}$$
$$+\mathrm{Re} < Fv - F_n v, F_n v - F_n v_n >.$$

Taking $n \to \infty$ *and* $F_n\, v \to F\, v$, we have

$$\| v - v_n \|^2 \leq \gamma \| v - v_n\|^2 + \eta \|v-v_n\|^2$$
$$\leq (\gamma+\eta) \|v-v_n\|^2,$$
$$\gamma+\eta<1.$$

So $v_n \to v$ *as* $n \to \infty$.

3. Coincidences Fixed Point Theorems

Our main results in this section are the following theorems

Theorem 3.1. Let H be a Hilbert space and let T, $f:H\to H$ be such that :

(i) $T(H)\subset f(H)$

$$(ii)\ \|Tx-Ty\|^2\le\alpha\ \max\left\{\begin{array}{l}\|fx-fx\|^2,\|fx-Tx\|^2,\\ \|fy-Ty\|^2,\\ \dfrac{\|fx-Ty\|^2+\|fy-Tx\|^2}{2}\end{array}\right\},\quad(3.1)$$

for all x , y $\in$ H where $0<\alpha<\frac{1}{8}$

(iii) f(H) or T(H) is a complete subspace of

(iv) T is asymptotically f - regular of some point x_0 in H.

Then T and f have a point of coincidence

Proof. Let $\{fx_n\}$ be an asymptotically T – regular sequence in H. Then by Parallelogram law we have

$$\begin{aligned}\|fx_n-fx_m\|^2&\le 2\|fx_n-Tx_n\|^2+2\|Tx_n-fx_m\|^2\\ &\le 2\|fx_n-Tx_n\|^2+4\|Tx_n-Tx_m\|^2\quad(3.2)\\ &\quad+4\|Tx_m-fx_m\|^2\end{aligned}$$

Using (3.1) we have

$$\begin{aligned}\|fx_n-fx_m\|^2&\le 2\ \|fx_n-Tx_n\|^2+2\ \|Tx_n-fx_m\|^2\\ &\le 2\ \|fx_n-Tx_n\|^2+\ 4\ \|Tx_m-fx_m\|\\ &\quad+4\ \alpha\max\left\{\begin{array}{l}\|fx_n-fx_m\|^2,\|fx_n-Tx_n\|^2,\\ \|fx_m-Tx_m\|^2,\\ \dfrac{\|fx_n-Tx_m\|^2+\|fx_m-Tx_n\|^2}{2}\end{array}\right\}\qquad(3.3)\end{aligned}$$

$$\begin{aligned}&\le 2\|fx_n-fx_{n+1}\|^2+\ 4\ \|fx_{m+1}-fx_m\|\\ &+4\ \alpha\max\left\{\begin{array}{l}\|fx_n-fx_m\|^2,\|fx_n-fx_{n+1}\|^2,\|fx_m-fx_{m+1}\|^2,\\ \dfrac{2\|fx_n-fx_m\|^2+2\|fx_m-fx_{m+1}\|^2}{2}+\\ \dfrac{2\|fx_m-fx_n\|^2+2\|fx_n-fx_{n+1}\|^2}{2}\end{array}\right\}.\end{aligned}$$

Taking limit as $n,m\to\infty$, and using asymptotically T – regular of $\{fx_n\}$ gives

$$\|fx_n-fx_m\|^2\le 4\ \alpha\max\left\{\|fx_n-fx_m\|^2,0,0,2\|fx_n-fx_m\|^2\right\}$$

$$\le 8\ \alpha\|fx_n-fx_m\|^2,\quad 0<\alpha<\frac{1}{8}$$

Hence we have

$$\lim_{n,m\to\infty}\|fx_n-fx_m\|=0.$$

It follows that $\{fx_n\}$ is a Cauchy sequence in H. If f(H) is a complete subspace of H, there exists a point $p\in H$ such that:

$fx_n\to p=fu$ (this also holds if T(H) is complete with $p\in T(H)$).

We claim that u is a coincidence point of f and T. If not $\|u-Tu\|>0$. From (ii), we obtain

$$\begin{aligned}\|fu-Tu\|^2&=\|p-fx_{n+1}+fx_{n+1}-Tu\|^2\\ &\le 2\|p-fx_{n+1}\|^2+2\|fx_{n+1}-Tu\|^2\\ &\le 2\|p-fx_{n+1}\|^2+2\|Tx_n-Tu\|^2\\ &\le 2\|p-fx_{n+1}\|^2\\ &\quad+2\alpha\max\left\{\begin{array}{l}\|fx_n-fu\|^2,\|fx_n-Tx_n\|^2,\|fu-Tu\|^2,\\ \dfrac{\|fx_n-Tx_n\|^2+\|fu-Tu\|^2}{2}\end{array}\right\}\end{aligned}$$

As $n\to\infty$ we get

$$\|fu-Tu\|^2=2\alpha\max\{0,0,\ \|fu-Tu\|^2,\ \frac{1}{2}\|fu-Tu\|^2\}.$$

Hence

$$\|fu-Tu\|^2\le\ 2\alpha\|fu-Tu\|^2,$$

a contradiction and so p = f u = T u is a point of coincidence of f and T.

Theorem 3.2. Let H be a Hilbert space and let T, $f:H\to H$ be such that:

(a) $T(H)\subset f(H)$

$$(b)\ \|Tx-Ty\|^2\le\alpha\ \max\left\{\begin{array}{l}\|fx-fy\|^2,\|fx-Tx\|^2,\\ \|fy-Ty\|^2,\\ \dfrac{\|fx-Ty\|^2+\|fy-Tx\|^2}{2}\end{array}\right\},\quad(3.4)$$

for all x , y $\in$ H where $0<\alpha<1$.

Then f, T have at least a unique point of coincidence.

Proof. Assume there exist p,p^* in H such that

P = f u = T u and $p^*=fu^*=Tu^*$ *for some* $u,u^*\in H$

From (3.4) we obtain

$$\begin{aligned}\|p-p^*\|^2&=\|Tu-Tu^*\|\\ &\le\alpha\ \max\left\{\begin{array}{l}\|fu-fu^*\|^2,\|fu-Tu\|^2,\|fu^*-Tu^*\|^2,\\ \dfrac{\|fu-Tu^*\|^2+\|fu^*-Tu\|^2}{2}\end{array}\right\}\\ &\le\alpha\ \max\left\{\|p-p^*\|^2,0,0,\frac{\|p-p^*\|^2+\|p^*-p\|^2}{2}\right\}\\ &\le\ \alpha\max\left\{\|p-p^*\|^2,0,0,\|p-p^*\|^2\right\}\le\alpha\|p-p^*\|^2,\end{aligned}$$

we deduce that $p^*=p$.

Remark. 3.1. If we put $f = Id$ (*identity map*) in Theorem 3.1and in Theorem 3.2, we obtain Theorems 3. 1and Theorem 3.2 in [1].

Let $F_i : [0,\infty) \to [0,\infty)$ be functions such that $F_i(0)$ and F_i is continuous at 0 (i=1,2). Ciric [5] studied necessary conditions to obtain a fixed point result of asymptotically regular mappings on complete metric spaces. M. Abbas and H.Aydi [1] extended the results of Ćirić [5] to the case of two mappings satisfying a generalized contractive conditions in a metric space and they proved the following theorem.

Theorem 3.3. **[5]**. Let (X, d) be a metric space. Let $T, f\colon X \to X$ be such that $T(X) \subset f(X)$. Assume that the following condition holds:

$$\begin{aligned} d(Tx,Ty) &\le a_1F_1[\min\{d(fx,Tx), f(fy,Ty)\}] \\ &+a_2F_2[d(fx,Tx)d(fy,Ty)] + a_3d(fx,fy) \\ &+a_4[d(fx,Tx)+d(fy,Ty)] + a_5[d(fx,Ty)+d(fy,Tx)], \end{aligned} \tag{3.5}$$

for all $x, y \in X$, *where* $i = 1,2,......5, a_i \ge 0$ such that for arbitrary fixed k > 0, $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$ and $a_1a_2 \le k$. If $f(X)$ or T(X) is a complete subspace of X and T is asymptotically f- regular at some point x_0 in X, then T and f have a unique point of coincidence.

In the following theorem we prove a similar result using a contractive condition (3.5) in a Hilbert space.

Theorem 3.4. Let H be a Hilbert space. Let $f, T : H \to H$ be such that:

(1) $T(H) \subset f(H)$

(2) $\|Tx-Ty\|^2 \le a_1F_1[\min\{\|fx-Tx\|^2, \|fy-Ty\|^2\}]+$
$a_2F_2[\|fx-Tx\|\|fy-Ty\|] + a_3\|fx-fy\|^2 + a_4[\|fx-Tx\|^2$
$+\|fy-Ty\|^2] + a_5[\|fx-Ty\|^2 + \|fy-Tx\|^2]$, (3.6)

for all $x, y \in H$, *where* $a_i \ge o, i = 1,2,....,5$ such that for arbitrary fixed $k > 0$, $0 < \lambda_1 < \frac{1}{2}$ *and* $0 < \lambda_2 < \frac{1}{8}$, we have $a_4 + a_5 \le \lambda_1$ $a_3 + a_5 < \lambda_2$ and $a_1a_2 < k$. If $f(H) \subset T(H)$ is a complete subspace of H and if T is asymptotically f – regular at some point x_0 *in* X. Then f, T have a point of coincidence.

Proof. Let x_0 be an arbitrary point in H and let $\{fx_n\}$ be a T-sequence with initial point x_0. Since T is asymptotically f-regular mappings at x_0 therefore $\|fx_n - fx_{n+1}\| \to 0$ *as* $n \to \infty$. Now for m > n, we have

$$\begin{aligned} \|fx_n - fx_m\|^2 &= \|(fx_n - Tx_n) + (Tx_n - fx_m)\|^2 \\ &\le 2\|fx_n - Tx_n\|^2 + 2\|Tx_n - fx_m\|^2 \\ &\le 2\|fx_n - Tx_n\|^2 + 4\|Tx_n - Tx_m\|^2 + 4\|Tx_m - fx_m\|^2 \\ &\le 2\|fx_n - fx_{n+1}\|^2 + 4\|fx_m - fx_{m+1}\|^2 \\ &+4a_1F_1[\min\{\|fx_n - fx_{n+1}\|^2, \|fx_m - fx_{m+1}\|^2\}] \\ &+4a_2F_2[\|fx_n - fx_{n+1}\|^2\|fx_m - fx_{m+1}\|^2] \\ &+4a_3\|fx_n - fx_m\|^2 + +4a_4[\|fx_n - fx_{n+1}\|^2 + \|fx_m - fx_{m+1}\|^2] \\ &+4a_5[\|fx_n - fx_m\|^2 + 2\|fx_m - fx_{m+1}\|^2 + 2\|fx_n - fx_{n+1}\|^2]. \end{aligned}$$

Thus we obtain that

$$\begin{aligned} &1-4(a_3+a_5)\|fx_n - fx_m\|^2 \\ &\le 2\|fx_n - fx_{n+1}\|^2 + 4\|fx_m - fx_{m+1}\|^2 + 4\|Tx_m - fx_m\|^2 \\ &+4a_1F_1[\min\{\|fx_n - fx_{n+1}\|^2, \|fx_m - fx_{m+1}\|^2\}] \\ &+4a_2F_2[\|fx_n - fx_{n+1}\|^2\|fx_m - fx_{m+1}\|^2] \\ &+4a_3\|fx_n - fx_m\|^2 \\ &+8a_5[\|fx_m - fx_{m+1}\|^2 + \|fx_n - fx_{n+1}\|^2]. \end{aligned} \tag{3.7}$$

Since T is asymptotically f –regular and F_1, F_2 are continuous at zero, then the right hand of the inequality (3.7) tends to zero as $m, n \to \infty$. Thus

$$\lim_{n,m\to\infty}\|fx_n - fx_m\| = 0.$$

It follows that $\{fx_n\}$ is a Cauchy sequence in H. If f(H) is a complete subsequence of H there exists u, p in H such that $fx_n \to p = fu$ (this holds also if T(H) is complete with $p \in T(H)$).

We claim that u is a coincidence point of f and T. If not $\|u - Tu\| > 0$. From (3.6) we obtain:

$$\begin{aligned} \|fu-Tu\|^2 &= \|p-Tu\|^2 \le 2\|p-fx_{n+1}\|^2 + 2\|fx_{n+1}-Tu\|^2 \\ &\le 2\|p-fx_{n+1}\|^2 + 2\|Tx_n - Tu\|^2 \\ &\le 2\|p-fx_{n+1}\|^2 + 2a_1F_1[\min\{\|fx_n - Tx_n\|^2, \|fu-Tu\|^2\}] \\ &+2a_2F_2[\|fx_n - Tx_n\|\|fu-Tu\|] + 2a_3\|fx_n - fu\|^2 \\ &+2a_4[\|fx_n - Tx_n\|^2 + \|fu-Tu\|^2] \\ &+2a_5[\|fx_n - Tu\|^2 + \|fu - Tx_n\|^2], \end{aligned}$$

which yields that

$$\begin{aligned} \|fu-Tu\|^2 &\le 2\|p-fx_{n+1}\|^2 \\ &+2a_1F_1[\min\{\|fx_n - fx_{n+1}\|^2, \|fu-Tu\|^2\}] \\ &+2a_2F_2[\|fx_n - fx_{n+1}\|\|fu-Tu\|] + 2a_3\|fx_n - fu\|^2 \\ &+2a_4[\|fx_n - fx_{n+1}\|^2 + \|fu-Tu\|^2] \\ &+2a_5[\|fx_n - Tu\|^2 + \|fu - fx_{n+1}\|^2]. \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\|fu-Tu\|^2 \le 2(a_4+a_5)\|fu-Tu\|^2 \le 2\lambda_1\|fu-Tu\|^2, 0 < \lambda_1 < \frac{1}{2},$$

a contradiction and so p = fu = T u is a point of a coincidence of f and T.

Lemma 3.5. Let H be a Hilbert space. Let $f, T : H \to H$ be such that:

(1) $T(H) \subset f(H)$

(2) $\|Tx-Ty\|^2 \le a_1F_1[\min\{\|fx-Tx\|^2, \|fy-Ty\|^2\}]$

$$+a_2F_2[\|fx-Tx\|\|fy-Ty\|] + a_3\|fx-fy\|^2 + a_4[\|fx-Tx\|^2 + \|fy-Ty\|^2] + a_5[\|fx-Ty\|^2 + \|fy-Tx\|^2],$$

for all $x, y \in H$, *where* $a_i \geq 0, i = 1,2,....,5$ such that for arbitrary fixed $k > 0$, $a_1 a_2 \leq k$ *and* $a_3 + 2a_5 < 1$. Then f, T have at most a unique coincidence point

Proof. Assume that there exists, p, p^* in H such that $p = fu = Tu$ *and* $p^* = fu^* = Tu^*$ *for some* $u, u^* \in H$ such that

$$\left\|p - p^*\right\|^2 = \left\|Tu - Tu^*\right\|^2$$
$$\leq a_1 F_1[\min\left\{\| fu - Tu\|^2, \left\| fu^* - Tu^*\right\|^2\right\}$$
$$+ a_2 F_2[\| fu - Tu\| \left\| fu^* - Tu^*\right\|] + a_3 \left\| fu - fu^*\right\|^2$$
$$+ a_4[\| fu - Tu\|^2 + \left\| fu^* - Tu^*\right\|^2] + a_5[\left\| fu - Tu^*\right\|^2 + \left\| fu^* - Tu\right\|^2].$$

Hence we obtain

$$\left\|p - p^*\right\|^2 \leq (a_3 + 2a_5)\left\|p - p^*\right\|^2, \quad a_3 + 2a_5 < 1.$$

This shows that $p = p^*$.

From Theorem 3.4 and Lemma 3.5 we obtain the following theorem

Theorem 3.6. Let H be a Hilbert space and let f, T be mappings on H into H such that $T(H) \subset f(H)$. Assume that T and f satisfy condition 3.6 for all $x, y \in H$. If f(H) or T(H) is a complete subspace of H such that (T, f) is weakly compatible, then T and f have a unique common fixed point provided that T is asymptotically f- regular at some point x_0 in H.

As a consequence of Theorem 3.4, Lemma 3.5 and Theorem 3.6 we obtain the following corollary.

Corollary 3.7. Let H be a Hilbert space. Let $T: H \to H$ be such that the following condition holds:

$$\begin{aligned} \|Tx - Ty\|^2 \leq\ & a_1 F_1[\min\left\{\|x - Tx\|^2, \|y - Ty\|^2\right\}] \\ & + a_2 F_2[\|x - Tx\|\|y - Ty\|] + a_3\|x - fy\|^2 \\ & + a_4[\|x - Tx\|^2 + \|y - Ty\|^2] \\ & + a_5[\|x - Ty\|^2 + \|y - Tx\|^2], \end{aligned} \tag{3.8}$$

for all $x, y \in H$, where $a_i \geq 0$ $(i = 1,2,...5)$ such that for an arbitrary fixed $k > 0$, $0 < \lambda_1 < \frac{1}{2}$, $0 < \lambda_2 < \frac{1}{4}$ we have $a_4 + a_5 < \lambda_1$, $a_3 + a_5 < \lambda_2$, $a_1 a_2 \leq k$. If T is asymptotically regular at some point x_0 in H. Then T has a unique fixed point.

Taking $a_1 = a_2 = 0$ in the inequality 3.6, we have tha following corollary.

Colorary 3.8. Let H be a Hilbert space. Let $f, T: H \to H$ be such that $T(H) \subset f(H)$. Assume that the following condition holds:

$$\|Tx - Ty\|^2 \leq a_3\| fx - fy\|^2 + a_4[\| fx - Tx\|^2 + \| fy - Ty\|^2]$$
$$+ a_5[\| fx - Ty\|^2 + \| fy - Tx\|^2],$$

for all $x, y \in H$, where $a_i \geq 0$ $(i = 3,...5)$ such that for arbitrary fixed $k > 0$, $0 < \lambda_1 < \frac{1}{2}$, $0 < \lambda_2 < \frac{1}{4}$, we have $a_4 + a_5 < \lambda_1$, $a_3 + a_5 < \lambda_2$, $a_1 a_2 \leq k$. If f(H) or T(H) is a complete subspace of H and if T is asymptotically regular at some point x_0 in H. Then T and f have a point. of coincidence.

We give an example to support our results

Example 3.9. Let X= $[0, \infty)$ and let Le t $T: H \to H$ be defined as

$$Tx = \frac{x}{4}, \quad fx = x.$$

Let x_0 =1 and the sequence $\{x_n\}_{n=1}^{n=\infty}$ be given by $x_n = \frac{1}{2^n}$. Note that $\{ fx_n \}$ is a T-sequence with initial point x_0. Since $\| f x_n - f x_{n+1}\| \to 0$ *as* $n \to \infty$, the mapping T is asymptotically f – regular at the point x_0. Also T(X)$\subset f(X)$, T(X) is a complete and the inequality (3.6) holds for all $x, y \in X$ with

$$F_1 = F_2 = 1, \; a_1 = a_2 = a_4 = a_5 = 0, \; a_3 = \frac{1}{16}.$$

Thus $f, T: X \to X$ satisfy all conditions of Theorem 3.4. Moreover u=0 is the common fixed point of f and T.

Let $CF(T, f)$ denote the set of all common fixed points of f and T. Now we have the following result on the continuity on the set of common fixed points

Theorem 3.10. Let H be a Hilbert space. Assume that $f, T: H \to H$ satisfy condition (3.6) for all $x, y \in H$. If $CF(T, f) \neq \phi$. Then T is continuous at $p \in CF(T, f)$ whenever f is continuous at p.

Proof. Fix $p \in CF(T, f)$. Let $\{z_n\}$ be any sequence in H converging to p. Then by taking $y := z_n$ and z:=p in (3.6) we get

$$\begin{aligned} \|Tx - Tz_n\|^2 \leq\ & a_1 F_1[\min\left\{\| fp - Tp\|^2, \| fz_n - Tz_n\|^2\right\}] \\ & + a_2 F_2[\| fp - Tp\|\| fz_n - Tz_n\|] + a_3\| fp - fz_n\|^2 \\ & + a_4[\| fp - Tp\|^2 + \| fz_n - Tz_n\|^2] \\ & + a_5[\| fp - Tz_n\|^2 + \| fz_n - Tp\|^2], \end{aligned}$$

which in view of T p = f p, we obtain

$$\|Tp - Tz_n\|^2 \leq a_3\| fp - fz_n\|^2 + a_4\|Tp - Tz_n\|^2$$
$$+ a_5[\|Tp - Tz_n\|^2 + \| fz_n - Tp\|^2].$$

Now by letting $n \to \infty$, we get

$$\overline{\lim_{n\to\infty}} \| Tp - Tz_n \|^2 \leq (a_4 + a_5) \overline{\lim_{n\to\infty}} \| Tp - Tz_n \|^2,$$

whenever f is continuous at p. The last inequality is true only if

$$\overline{\lim_{n\to\infty}} \|Tp - Tz_n\|^2 = 0.$$

We get that $Tz_n \to Tp \quad as \quad n \to \infty$.

4. Well – Posedness

The notion of well - posedness of a fixed point problem has generated much interest to several mathematicians, for example [2,3,8,11,12,13]. Here, we study well – posedness of a common fixed point problem.

Definition 4.1. Let H be a Hilbert space and $f: H \to H$ be a mapping. The fixed point problem of f is said to be well posed if

(i) f has a unique fixed point x_0 in H

(ii) for any sequence $\{x_n\} \in H$, $\lim_{n\to\infty} \|x_n - fx_n\| = 0$ we have $\lim_{n\to\infty} \|x_n - x_0\| = 0.$

Definition 4.2. A common fixed point problem of self-maps f and T on H, CFP(f,T,H) is called well- posed if CF(f, T) (the set of all common fixed points of f and T) is singletion and for any sequence $\{x_n\}$ in H with $x^* \in CF(f,T)$ and $\lim_{n\to\infty} \| fx_n - x_n\| = \lim_{n\to\infty} \| Tx_n - x_n\| = 0$ implies $x^* = \lim_{n\to\infty} x_n$.

Theorem 4.1. Suppose that T and f be self –maps on H as in Theorem 3.4 and Lemma 3.1. Then the common fixed problem of f and T is well posed.

Proof. From Theorem 3.4 and Lemma 3.1, the mappings f and T have a unique common fixed point, say u $\in H$. Let $\{x_n\}$ be a sequence in H and $\lim_{n\to\infty} \| fx_n - x_n\| = \lim_{n\to\infty} \| Tx_n - x_n\| = 0$. With loss of generality, we may suppose that $u \neq x_n$ for every nonnegative integer n. Then having in mind fu =T u and from triangle inequality (3.6), we have,

$$\|u - x_n\|^2 = \|Tu - x_n\|^2 \le 2\|Tu - Tx_n\|^2 + 2\|Tx_n - x_n\|^2$$
$$\le 2\|Tx_n - x_n\|^2 + a_1F_1[\min\left\{\| fx_n - Tx_n\|^2 , \| fu - Tu\|^2\right\}]$$
$$+ a_2F_2[\| fx_n - Tx_n\|\| fu - Tu\|] + a_3 \| fx_n - fu\|^2$$
$$+ a_4[\| fx_n - Tx_n\|^2 + \| fu - Tu\|^2] + a_5[\| fx_n - Tu\|^2 + \|u - Tx_n\|^2]$$
$$\le 2\|Tx_n - x_n\|^2 + a_3[2\| fx_n - x_n\|^2 + 2\|x_n - u\|^2]$$
$$+ a_4[2\| fx_n - x_n\|^2 + 2\|x_n - Tx_n\|^2] + a_5[2\| fx_n - x_n\|^2$$
$$+ 4\|x_n - u\|^2 + 2\|x_n - Tx_n\|^2].$$

Letting $n \to \infty$, we get that $\overline{\lim_{n\to\infty}} \| u - x_n\| = 0$. We deduce, $x_n \to u \quad as \quad n \to \infty$. This completes the proof of Theorem.

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