American Journal of Mathematical Analysis, 2014, Vol. 2, No. 1, 4-7
Available online at http://pubs.sciepub.com/ajma/2/1/2

DOI:10.12691/ajma-2-1-2

On Inequalities of Trigonometrically ρ- Convex Functions

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Received January 19, 2014; Revised February 13, 2014; Accepted February 17, 2014

Abstract The main goal of this paper is to derive two integral inequalities for trigonometrically ρ - convex functions which are closely connected with Andersson's inequality for ordinary convex functions.

Keywords: *generalized convex functions, trigonometrically ρ - convex functions, supporting functions, integral inequalities*

Cite This Article: Mohamed Sabri Salem Ali, "On Inequalities of Trigonometrically ρ- Convex Functions." *American Journal of Mathematical Analysis*, vol. 2, no. 1 (2014): 4-7. doi: 10.12691/ajma-2-1-2.

1. Introduction and Preliminaries

Trigonometrically ρ - convex functions play a central role in many various aspects in the theory of entire functions (of order $0<\rho<\infty$) and in the theory of cavitational diagrams for hydroprofiles, see for example [4,5,6]. In what follows, we shall be concerned with real finite functions defined on a finite or infinite interval $I\subset\mathbb{R}$. In this section we present the basic definitions and results which will be used later, for more information see [5,6].

Definition 1.1. A function $f:I\to\mathbb{R}$ is said to be trigonometrically ρ - convex, if for any arbitrary closed subinterval $[u,v]$ of I such that $0<\rho(v-u)<\pi$, the graph of $f(x)$ for $x\in[u,v]$ lies nowhere above the ρ - trigonometric function, determined by the equation:

$$H(x)=H(x;u,v,f)=A\cos\rho x+B\sin\rho x\ ,$$

where A and B are chosen such that

$$H(u)=f(u),\ \text{and}\ H(v)=f(v).$$

Equivalently, if for all $x\in[u,v]$

$$f(x)\le H(x)=\frac{f(u)\sin\rho(v-x)+f(v)\sin\rho(x-u)}{\sin\rho(v-u)}.$$

Definition 1.2. A function

$$T_u(x)=A\cos\rho x+B\sin\rho x$$

is said to be supporting function for $f(x)$ at the point $u\in I$, if

$$T_u(u)=f(u),\quad and\quad T_u(x)\le f(x)\quad \forall x\in I.$$

That is, if $f(x)$ and $T_u(x)$ agree at $x=u$, and the graph of $f(x)$ does not lie under the support curve.

Theorem 1.1. [1] A function $f:I\to\mathbb{R}$ is trigonometrically ρ - convex on I if and only if there exists a supporting function for $f(x)$ at each point $x\in I$.

Property 1.1. [1] If $f:I\to\mathbb{R}$ is differentiable trigonometrically ρ - convex function, then the supporting function for $f(x)$ at the point $u\in I$ has the form:

$$T_u(x)=f(u)\cos\rho(x-u)+f'(u)\sin\rho(x-u).$$

To keep the paper self contained we present the following results [7].

Theorem 1.2. If $f:(a,b)\to\mathbb{R}$ is differentiable function. Then f is convex if and only if f' is increasing.

Theorem 1.3. A function $f:(a,b)\to\mathbb{R}$ is convex if and only if there is an increasing function $g:(a,b)\to\mathbb{R}$ and a point $c\in(a,b)$ such that for all $x\in(a,b)$,

$$f(x)-f(c)=\int_c^x g(t)\,dt.$$

Theorem 1.4. If $f:(a,b)\to\mathbb{R}$ and $g:(a,b)\to\mathbb{R}$ are both non-negative, increasing (decreasing), and convex, then $h(x)=f(x)g(x)$ also preserves these three properties.

In [3], B. J. Andersson established the following theorem

Theorem 1.5. Let $F_1(x),F_2(x),\dots,F_n(x)$ be convex functions, defined in $0\le x\le 1$, and for which

$$F_p(x)\ge 0,\quad F_p(0)=0,\quad p=1,2,\dots,n.$$

If $\int_0^1 F_p(x)dx = \alpha_p$, then

$$\int_0^1 F_1(x)F_2(x)...F_n(x)dx \geq \frac{2^n}{n+1}\alpha_1\ \alpha_2 ...\ \alpha_n. \qquad (1)$$

2. Results

In this section we derive a similar result to Andersson's inequality for trigonometrically ρ - convex functions. For more inequalities, one may refer to [2].

Theorem 2.1. Let $g_1(x), g_2(x), ..., g_n(x)$ be continuous convex functions, defined in $0 \leq x \leq \frac{\pi}{2\rho}$, for which

$$g_k(x) \geq 0, \quad g_k(0) = 0, \quad k = 1, 2, ..., n,$$

and let $f(x)$ be a non-negative, $\frac{2\pi}{\rho}$ - periodic, differentiable, and trigonometrically ρ - convex function defined on $\mathbb{R}$, such that: $f'(0) = 0$ and $\int_0^{\frac{\pi}{2\rho}} g_k(x)\cos\rho x\ dx = \alpha_k$,

then

$$\int_0^{\frac{\pi}{2\rho}} f(x)\prod_{k=1}^n g_k(x)\,dx \geq \frac{2^n f(0)}{\rho(n+1)}\prod_{k=1}^n \alpha_k.$$

Proof. As $f(x)$ is trigonometrically ρ - convex function, then from Definition 1.2, it follows that:

$$f(x) \geq T_0(x) \qquad \forall x \in [0, \frac{\pi}{2\rho}].$$

Since $f(x)$ is differentiable and $f'(0) = 0$, then from Property 1.1, the supporting function $T_0(x)$ for $f(x)$ at the point $0 \in [0, \frac{\pi}{2\rho}]$ can be written in the form

$$T_0(x) = f(0)\cos\rho x.$$

Consequently,

$$f(x) \geq f(0)\cos\rho x \qquad \forall x \in [0, \frac{\pi}{2\rho}]. \qquad (2)$$

As $\prod_{k=1}^n g_k(x) \geq 0$, by using (2), one has:

$$\int_0^{\frac{\pi}{2\rho}} f(x)\prod_{k=1}^n g_k(x)dx \geq f(0)\int_0^{\frac{\pi}{2\rho}}\prod_{k=1}^n g_k(x)\cos\rho x\,dx. \quad (3)$$

Using the following substitution

$$t = \sin\rho x, \qquad (4)$$

and let

$$h_k(t) = g_k(x), \qquad (5)$$

then it follows that:

$$h_k(0) = 0, \quad h_k(t) \geq 0 \quad \forall t \in [0,1],$$

and $h_k'(t)$ is an increasing function in [0,1]. Using Theorem 1.2, one obtains that $h_k(t)$ is a convex function in [0,1].

Let

$$\int_0^1 h_k(t)dt = \alpha_k, \quad k = 1, 2, ..., n,$$

we obseve that the functions $h_k(t)$, $k = 1, 2, ..., n$ satisfy all assumptions of the Theorem 1.5 in the interval [0,1].

Hence, using (1), one obtains:

$$\int_0^1 \prod_{k=1}^n h_k(t)\,dt \geq \frac{2^n}{n+1}\prod_{k=1}^n \alpha_k. \qquad (6)$$

Now using (4), (5), and (6), then (3) turns out to:

$$\int_0^{\frac{\pi}{2\rho}} f(x)\prod_{k=1}^n g_k(x)dx \geq \frac{f(0)}{\rho}\int_0^1\prod_{k=1}^n h_k(t)\,dt \geq \frac{2^n f(0)}{\rho(n+1)}\prod_{k=1}^n \alpha_k,$$

where $\alpha_k = \int_0^1 h_k(t)dt = \int_0^{\frac{\pi}{2\rho}} g_k(x)\cos\rho x\ dx$.

Hence, the claim.

Now, we are in a position to prove the following main theorem.

Theorem 2.2. Let $g_1(x), g_2(x), ..., g_n(x)$ be convex functions, defined in $0 \leq x \leq \frac{\pi}{2\rho}$, for which

$$g_r(x) \geq 0, \quad g_r(0) = 0, \quad r = 1, 2, ..., n,$$

and let $f(x)$ be a non-negative, $\frac{2\pi}{\rho}$ - periodic, differentiable, and trigonometrically ρ - convex function defined on $\mathbb{R}$, such that: $f'(\pi/2\rho) = 0$ and $\int_0^{\frac{\pi}{2\rho}} g_r(x)\,dx = \alpha_r$.

Then, one has the following sharp inequality:

$$\int_0^{\frac{\pi}{2\rho}} f(x)\prod_{r=1}^n g_r(x)\,dx \geq \frac{\pi}{(n+2)} f(\pi/2\rho)\left(\frac{4\rho}{\pi}\right)^{n-1} \times \left(\prod_{r=1}^n \alpha_r\right) {}_1F_2(1+\frac{n}{2}; \frac{3}{2}, 2+\frac{n}{2}; -\frac{\pi^2}{16}). \qquad (7)$$

Proof. The proof will be divided into 4 steps.

Let M denote the class of convex functions of the theorem.

Step 1. If $g \in M$, *then* g *is increasing.*

Since $g(x)$ is convex in $[0, \frac{\pi}{2\rho}]$ and $g(0) = 0$, then from Theorem 1.3 there is an increasing function $h: [0, \frac{\pi}{2\rho}] \to \mathbb{R}$

such that

$$g(x) = \int_0^x h(t)dt, \quad x \in [0, \frac{\pi}{2\rho}]. \tag{8}$$

Now, suppose that $h(t_0) < 0$ for some $t_0 \in [0, \frac{\pi}{2\rho}]$.

As h is increasing, then $h(t) < 0$ for all $t \in [0, t_0]$, therefore, $\int_0^x h(t)dt < 0, \quad x \in [0, t_0]$.

From (8) it follows, $g(x) < 0, \quad x \in [0, t_0]$, which contradicts the fact that $g(x) \geq 0, \quad x \in [0, \frac{\pi}{2\rho}]$.

Thus, $h(t) \geq 0$ for all $t \in [0, \frac{\pi}{2\rho}]$.

Now, let $x_1, x_2 \in [0, \frac{\pi}{2\rho}]$, if $x_1 < x_2$, then using (8), one has:

$$0 \leq \int_{x_1}^{x_2} h(t)dt = \int_0^{x_2} h(t)dt - \int_0^{x_1} h(t)dt = g(x_2) - g(x_1).$$

Hence, $g(x)$ is an increasing function.

For the next steps, let

$$g_r^*(x) = \frac{2\alpha_r x}{(\pi/2\rho)^2}, \quad 0 \leq x \leq \frac{\pi}{2\rho}, \tag{9}$$

and

$$\Phi_r(x) = \int_0^x \left[g_r^*(s) - g_r(s)\right]ds. \tag{10}$$

Step 2. M is closed under multiplication.

From (9), it follows that:

$g_r^*(x)$ is non-negative, increasing, and convex function satisfies $g_r^*(0) = 0$.

Thus, $g_r^* \in M, \quad r = 1, 2, ..., n$.

Hence, from Theorem 1.4, M is closed under multiplication.

This confirm that this property is independent of the choice of g_r's or g_r^*'s.

Step 3. $\Phi_r(x) \geq 0, \quad x \in [0, \frac{\pi}{2\rho}]$.

Using (9) and (10), one concludes that

$$\int_0^{\frac{\pi}{2\rho}} g_r^*(x)\,dx = \int_0^{\frac{\pi}{2\rho}} g_r(x)\,dx = \alpha_r, \tag{11}$$

$$\Phi_r(0) = 0, \text{ and } \Phi_r(\pi/2\rho) = 0. \tag{12}$$

From the convexity of $g_r(x)$, it follows that the graph of $g_r(x)$ must intersect the straight line of $g_r^*(x)$ in a unique point q as shown in Figure 1.

If x lies in $[0, p]$, then obviously from (10) it followos that

$$\Phi_r(x) \geq 0, \quad 0 \leq x \leq p.$$

Otherwise, if $p \leq x \leq \frac{\pi}{2\rho}$, one has:

$$\int_x^{\frac{\pi}{2\rho}} g_r(s)ds \geq \int_x^{\frac{\pi}{2\rho}} g_r^*(s)ds.$$

Using (11), one obtains:

$$\int_0^{\frac{\pi}{2\rho}} g_r^*(s)ds - \int_x^{\frac{\pi}{2\rho}} g_r^*(s)ds \geq \int_0^{\frac{\pi}{2\rho}} g_r(s)ds - \int_x^{\frac{\pi}{2\rho}} g_r(s)ds$$

$$\int_0^x \left[g_r^*(s) - g_r(s)\right]ds \geq 0.$$

Thus, $\Phi_r(x) \geq 0, \quad p \leq x \leq \frac{\pi}{2\rho}$.

Hence $\Phi_r(x) \geq 0, \quad x \in [0, \frac{\pi}{2\rho}]$.

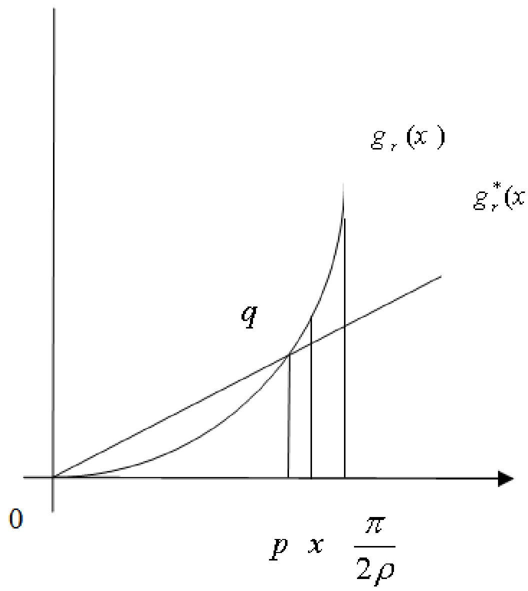

Figure 1.

As $f(x)$ is differentiable trigonometrically ρ -convex function and $f'(\pi/2\rho) = 0$, according to Property 1.1, for convenience, we denote

$$f^*(x) = f(\pi/2\rho)\sin\rho x, \tag{13}$$

as the supporting function for $f(x)$ at the point $\frac{\pi}{2\rho} \in [0, \frac{\pi}{2\rho}]$.

But, from Definition 1.2, one obtains:

$$f(x) \geq f^*(x), \quad x \in [0, \frac{\pi}{2\rho}], \tag{14}$$

hence we can go to:

Step 4. We show that If $G_1(x), G_2(x) \in M$, then

$$\int_0^{\frac{\pi}{2\rho}} G_1(x)G_2(x)f(x)dx \geq \int_0^{\frac{\pi}{2\rho}} G_1^*(x)G_2(x)f^*(x)dx. \tag{15}$$

From *Step 2*, we observe:

$$G_1(x)G_2(x) \geq 0, \quad x \in [0, \frac{\pi}{2\rho}].$$

Using (14), one has,

$$\int_0^{\frac{\pi}{2\rho}} G_1(x)G_2(x)f(x)dx \ge \int_0^{\frac{\pi}{2\rho}} G_1(x)G_2(x)f^*(x)dx. \quad (16)$$

Let

$$S_1(x) = G_2(x)f^*(x). \quad (17)$$

Using (10) and (12), it follows that

$$\int_0^{\frac{\pi}{2\rho}} G_1(x)S_1(x)dx = \int_0^{\frac{\pi}{2\rho}} S_1(x)[G_1^* - (G_1^* - G_1)](x)dx$$

$$= \int_0^{\frac{\pi}{2\rho}} S_1(x)G_1^*(x)dx - \int_0^{\frac{\pi}{2\rho}} S_1(x)\, d\Phi_1(x)$$

$$= \int_0^{\frac{\pi}{2\rho}} G_1^*(x)S_1(x)dx + \int_0^{\frac{\pi}{2\rho}} \Phi_1(x)\, dS_1(x).$$

Since $dS_1(x) \ge 0$, and from *Step 3,* we infer that

$$\int_0^{\frac{\pi}{2\rho}} \Phi_1(x)dS_1(x) \ge 0.$$

Thus

$$\int_0^{\frac{\pi}{2\rho}} G_1(x)S_1(x)dx \ge \int_0^{\frac{\pi}{2\rho}} G_1^*(x)S_1(x)dx. \quad (18)$$

Hence, from (16), (17) and (18), we obtain the required inequality (15).

Now, we prove the main inequality (7).

From *Step 2,* we have

$$g_n(x),\ g_1(x)g_2(x)\ldots g_{n-1}(x) \in M.$$

Thus, using (15), one has

$$\int_0^{\frac{\pi}{2\rho}} g_1(x)\ldots g_{n-1}(x)g_n(x)f(x)dx \ge$$

$$\int_0^{\frac{\pi}{2\rho}} g_1(x)\ldots g_{n-1}(x)g_n^*(x)f^*(x)dx.$$

Again, $g_{n-1}(x),\ g_1(x)\ldots g_{n-2}(x)g_n^*(x) \in M$.

Hence, from (18) it follows that

$$\int_0^{\frac{\pi}{2\rho}} g_1(x)\ldots g_{n-1}(x)g_n^*(x)f^*(x)dx \ge$$

$$\int_0^{\frac{\pi}{2\rho}} g_1(x)\ldots g_{n-1}^*(x)g_n^*(x)f^*(x)dx.$$

Repeating the above argument and using (18) each time, then from (9) and (13) one obtains:

$$\int_0^{\frac{\pi}{2\rho}} g_1(x)\ldots g_n(x)f(x)dx \ge \int_0^{\frac{\pi}{2\rho}} g_1^*(x)\ldots g_n^*(x)f^*(x)dx$$

$$= \frac{2^n f(\pi/2\rho)}{(\pi/2\rho)^{2n}}\left(\prod_{r=1}^{n}\alpha_r\right)\int_0^{\frac{\pi}{2\rho}} x^n \sin\rho x\, dx.$$

Using the following substitution

$$z = \rho x,$$

we get

$$\int_0^{\frac{\pi}{2\rho}} f(x)\prod_{r=1}^{n} g_r(x)dx \ge$$

$$\rho^n\left(\frac{8\rho}{\pi^2}\right)^n f(\pi/2\rho)\left(\prod_{r=1}^{n}\alpha_r\right)\frac{1}{\rho^{n+1}}\int_0^{\frac{\pi}{2}} z^n \sin z\, dz$$

$$= \frac{1}{\rho} f(\pi/2\rho)\left(\frac{8\rho}{\pi^2}\right)^n\left(\prod_{r=1}^{n}\alpha_r\right)\times$$

$$\frac{(\pi/2)^{n+2}}{(n+2)}\ {}_1F_2\left(1+\frac{n}{2};\frac{3}{2},2+\frac{n}{2};-\frac{\pi^2}{16}\right)$$

$$= \frac{\pi}{(n+2)} f(\pi/2\rho)\left(\frac{4\rho}{\pi}\right)^{n-1}\left(\prod_{r=1}^{n}\alpha_r\right)\times$$

$${}_1F_2\left(1+\frac{n}{2};\frac{3}{2},2+\frac{n}{2};-\frac{\pi^2}{16}\right),$$

and the theorem is proved.

Note: the equality in (7) occurs if

$$f(x) = f(\pi/2\rho)\sin\rho x,$$

$$g_r(x) = \frac{2\alpha_r x}{(\pi/2\rho)^2}, \qquad r = 1,2,\ldots,n.$$

Acknowledgement

The author wishes to thank the anonymous referees for their fruitful comments and suggestions which improved the original manuscript.

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