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Summability of a Jacobi Series by Lower Triangular Matrix Method

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Abstract The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ which is obtained from Jacobi differential equation is an orthogonal polynomial over the interval [-1, 1] with respect to weight function $(1-x)^{\alpha}(1+x)^{\beta}$, $\alpha>-1$, $\beta>-1$. Here Jacobi series has been taken and established a theorem on lower triangular matrix summability of a Jacobi series.

Keywords: *summability, jacobi series, triangular matrix*

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1. Introduction

The Nörlund summability (N, p_n) on Jacobi series has been studied by a number of researchers like Gupta [4], Choudhary [3], Thorpe [16], Pandey and Beohar [10], Prasad and Saxena [11], Beohar and Sharma [1], Pandey [9], Tripathi et al. [18] and Chandra [14]. After quite a good amount of work in the ordinary Nörlund summability of Jacobi series at the point x =1, Khare and Tripathi [5] discussed the generalized Nörlund summability (N, p, q) of Jacobi series. The (N, p, q) summability reduces to the (N, p_n) summability for $q_n = 1\ \forall n$. The Cesàro Summability of Jacobi series has been studied by Szili &Weisz [15]. The Cesàro Summability, Nörlund Summability, generalized Nörlund Summability are special cases of The matrix Summability method. In this paper a more general result than those Gupta [4], Choudhary [3], Khare and Tripathi [5] has been obtained so that their results come out as particular cases.

2. Definitions and Notations

Let f (x) be defined in closed interval [-1, 1] such that the function

$$(1-x)^{\alpha}(1+x)^{\beta} f(x) \in L[-1,1];\ \alpha>-1,\ \beta>-1.$$

The Jacobi series corresponding to this function is

$$\sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \qquad (2.1)$$

where

$$a_n = \frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x) P_n^{(\alpha,\beta)}(x)\,dx$$

and $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials.

Let $T=(a_{n,k})$ be an infinite lower triangular matrix method T satisfying the Silverman- Töeplitz [17] conditions of regularity i.e.

$$\sum_{k=0}^{n} a_{n,k} \to 1 \ as\ n\to\infty,$$

$a_{n,k}=0$, for k > n and $\sum_{k=0}^{n} \left|a_{n,k}\right| \le M$, where M is a finite positive constant.

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series whose n^{th} partial sum is given by

$$s_n = \sum_{\nu=0}^{n} u_{\nu}.$$

The sequence - to - sequence transformation

$$t_n = \sum_{k=0}^{n} a_{n,k}\, s_k$$

defines the sequence $\{t_n\}$ of matrix means of sequence $\{s_n\}$, generated by the sequence of coefficient $(a_{n,k})$.

If

$$t_n \to s \ as\ n\to\infty,$$

then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{s_n\}$ is said to be summable by matrix method to s. It is denoted by

$$t_n \to s(T)\ as\ n\to\infty \text{ (Zygumund [19]).}$$

We use the following notations:

$$F(\varphi)=\{f(\cos\varphi)-A\}\left[\left(\sin\frac{\varphi}{2}\right)^{2\alpha+1}\left(\cos\frac{\varphi}{2}\right)^{2\beta+1}\right] \qquad (2.2)$$

A being fixed constant.

$$\Psi(t)=\int_{o}^{t}|F(\varphi)|d\varphi$$

$$\tau = Integral\ \ part\ of\ \tfrac{1}{\varphi} = [\tfrac{1}{\varphi}]$$

3. Main Theorem

The purpose of this paper is to establish a theorem under a very general condition so that it generalizes all the known results for Nörlund summability (N,p_n) of Jacobi series in this direction. In fact, we prove the following:

Theorem: Let T = ($a_{n,k}$) be an infinite lower triangular regular matrix such that the element ($a_{n,k}$) is positive, monotonic increasing in k for 0 ≤ k ≤ n,

$$A_{n,\tau}=\sum_{k=n-\tau}^{n} a_{n,k},\quad A_{n,n}=1\,\forall n \text{ and}$$

$$n^{\alpha+\frac{1}{2}}A_{n,\left[\frac{1}{\delta}\right]}=o(1),\ 0<\delta<\pi \quad as\ \ n\to\infty.$$

If

$$\int_{1-t}^{1}|f(u)-A|du=o\left(\frac{t}{\xi(\frac{1}{t})\log\frac{1}{t}}\right)\ \ as\ \ t\to 0 \qquad (3.1)$$

then the Jacobi series (2.1) is summable (T) to the sum A at x = 1 provided $\xi(t)$ is positive monotonic non-decreasing function of t such that

$$\sum_{a}^{n}\frac{A_{n,k}}{k^{\frac{2\alpha+3}{2}}\xi(k)\log k}=O\left(\frac{1}{n^{\frac{2\alpha+1}{2}}}\right), \qquad (3.2)$$

$-\frac{1}{2}\le\alpha\le\frac{1}{2}$, $\beta>-\frac{1}{2}$ and the antipole condition

$$\int_{0}^{\frac{1}{n}} t^{\beta-\frac{1}{2}}|f(-\cos t)-A|dt=o(1)\ \ as\ n\to\infty \qquad (3.3)$$

is satisfied.

4. Lemmas

The following lemmas are required for the proof of the theorem:

Lemma 4.1. (Szegö, [13]): If α > -1, β > -1 then as $n\to\infty$

$$P_n^{(\alpha,\beta)}(\cos\varphi)=\begin{cases} O(n^{\alpha}) & \text{for } 0\le\varphi<\frac{1}{n} \quad (4.1.1)\\ O(n^{\beta}) & \text{for } \pi-\frac{1}{n}\le\varphi\le\pi \quad (4.1.2)\\ n^{-\frac{1}{2}}k(\varphi)\left[\cos(N\varphi+\nu)+\frac{O(1)}{n\sin\varphi}\right] & \text{for } \frac{1}{n}\le\varphi<\pi-\frac{1}{n} \quad (4.1.3)\end{cases}$$

where

$$k(\varphi)=\pi^{-\frac{1}{2}}\left(\sin\frac{\varphi}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{\varphi}{2}\right)^{-\beta-\frac{1}{2}}$$

$$N=n+\frac{\alpha+\beta+1}{2},\quad \nu=-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{4}.$$

Lemma 4.2. (Gupta, [4]): The antipole condition (.3.3) includes

$$\int_{-1}^{b}(1+x)^{\frac{2\beta-3}{4}}|f(x)|dx<\infty, \qquad (4.2.1)$$

b fixed, and

$$\int_{\delta}^{\pi}\left(\cos\frac{\varphi}{2}\right)^{\beta-\frac{1}{2}}|f(\cos\varphi)-A|d\varphi<\infty. \qquad (4.2.2)$$

Lemma 4.3 Condition (3.1) is equivalent to

$$\int_{0}^{t}|F(\varphi)|d\varphi=o\left(\frac{t^{2\alpha+2}}{\xi(\frac{1}{t})\log\frac{1}{t}}\right)\ ,\ \ as\quad t\to 0. \qquad (4.3.1)$$

Proof:

$$\int_{0}^{t}|F(\varphi)|d\varphi$$

$$=\int_{0}^{t}|f(\cos\varphi)-A|\left|\left(\sin\frac{\varphi}{2}\right)^{2\alpha+1}\right|\left|\left(\cos\frac{\varphi}{2}\right)^{2\beta+1}\right|d\varphi$$

$$\le\int_{0}^{t}|f(\cos\varphi)-A|(\tfrac{\varphi}{2})^{2\alpha+1}d\varphi$$

$$\le t^{2\alpha+1}\int_{0}^{t}|f(\cos\varphi)-A|d\varphi$$

$$=t^{2\alpha+1}\,o\left(\frac{t}{\xi(\frac{1}{t})\log\frac{1}{t}}\right)$$

$$=o\left(\frac{t^{2\alpha+2}}{\xi(\frac{1}{t})\log\frac{1}{t}}\right)$$

Conversely

$$\int_{0}^{t}|F(\varphi)|d\varphi\le t^{2\alpha+1}\int_{0}^{t}|f(\cos\varphi)-A|d\varphi$$

$$o\left(\frac{t^{2\alpha+2}}{\xi(\frac{1}{t})\log\frac{1}{t}}\right)=t^{2\alpha+1}\int_{1}^{1+t}|f(\cos\varphi)-A|d\varphi$$

$$o\left(\frac{t}{\xi(\frac{1}{t})\log\frac{1}{t}}\right)=\int_{1-t}^{1}|f(\cos\varphi)-A|d\varphi$$

$$\int_{1-t}^{1}|f(u)-A|du=o\left(\frac{t}{\xi(\frac{1}{t})\log\frac{1}{t}}\right).$$

Lemma 4.4 If ($a_{n,k}$) is non-negative and non-decreasing with 0 ≤ k ≤ n, then, for 0 ≤ a < b≤ ∞, 0 ≤ t ≤ π and for any n,

$$\left|\sum_{k=a}^{b} a_{n,k}\, e^{ikt}\right| = O(A_{n,\tau}) \qquad (4.4.1)$$

where $\tau = Integral\ \ part\ \ of\ \frac{1}{t} = [\frac{1}{t}]$.

Lemma 4.4 may be proved by the following technique of Lemma 4.1 in Lal [6].

Lemma 4.5 Under the condition of the theorem on $(a_{n,k})$, for large n, uniformly in $0 < \varphi \le \pi$, $0 \le a \le b \le n$,

$$\left|\sum_{k=a}^{b} a_{n,k} \cos\{(k+\rho)\varphi - \gamma\}\, k^{\alpha+\frac{1}{2}}\right| = O(n^{\alpha+\frac{1}{2}} A_{n,\tau}) \qquad (4.5.1)$$

where

$$\rho = \frac{\alpha+\beta+2}{2}, \quad \gamma = -\left(\alpha+\frac{3}{2}\right)\frac{\pi}{4}.$$

Proof:

$$\left|\sum_{k=a}^{b} a_{n,k} \cos\{(k+\rho)\varphi - \gamma\}\, k^{\alpha+\frac{1}{2}}\right|$$

$$= O(n^{\alpha+\frac{1}{2}})\left| \text{Re}\,al\ part\ of \sum_{k=a}^{b} a_{n,k}\, e^{i\{(k+\rho)\varphi-\gamma\}}\right|$$

$$= O(n^{\alpha+\frac{1}{2}})\left|\sum_{k=a}^{b} a_{n,k}\, e^{ik\varphi} e^{i(\rho\varphi-\gamma)}\right|$$

$$= O(n^{\alpha+\frac{1}{2}})\left|\sum_{k=a}^{b} a_{n,k}\, e^{ik\varphi}\right|$$

$$= O(n^{\alpha+\frac{1}{2}} A_{n,\tau}),$$

by Abel's Lemma.

Lemma 4.6 Under the hypothesis of the theorem,

$$\sum_{k=1}^{n} a_{n,k}\, k^{\alpha-\frac{1}{2}} = O(n^{\alpha-\frac{1}{2}}) \qquad (4.6.1)$$

Proof:

$$\sum_{k=1}^{n} a_{n,k}\, k^{\alpha-\frac{1}{2}} = \sum_{k=1}^{[n/2]} a_{n,k}\, k^{\alpha-\frac{1}{2}} + \sum_{k=[n/2]+1}^{n} a_{n,k}\, k^{\alpha-\frac{1}{2}}$$

$$= O(a_{n,[n/2]}) \sum_{k=1}^{[n/2]} k^{\alpha-\frac{1}{2}} + O(n^{\alpha-\frac{1}{2}}) \sum_{k=[n/2]+1}^{n} a_{n,k}$$

$$= O(a_{n,[n/2]})\, n^{\alpha+\frac{1}{2}} + O(n^{\alpha-\frac{1}{2}}),$$

since $\sum_{k=1}^{n} a_{n,k} = 1.$ Also, $1 \ge \sum_{k=[n/2]+1}^{n} a_{n,k} \ge \frac{n}{2} a_{n,[n/2]}$ and putting this in the above gives the result

$$\sum_{k=1}^{n} a_{n,k}\, k^{\alpha-\frac{1}{2}} = O(n^{\alpha-\frac{1}{2}}).$$

Lemma 4.7 Let

$$M_n(\varphi) = 2^{\alpha+\beta+1} \sum_{k=0}^{n} a_{n,k}\, \lambda_k\, P_k^{\alpha+1,\beta}(\cos\varphi)$$

where

$$\lambda_n = \frac{2^{-(\alpha+\beta+1)}\,\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(n+\beta+1)} \approx \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)} n^{\alpha+1}$$

then for $-\frac{1}{2} \le \alpha < \frac{1}{2}$, $\beta > -\frac{1}{2}$ and if $a_{n,k}$ satisfies the hypothesis of the theorem,

$$M_n(\varphi) = \begin{cases} O(n^{2\alpha+2}) & \text{for } 0 \le \varphi < \frac{1}{n} \quad (4.7.1) \\ O(n^{\alpha+\beta+1}) & \text{for } \pi - \frac{1}{n} \le \varphi \le \pi \quad (4.7.2) \\ O\left(n^{\alpha+\frac{1}{2}} A_{n,\tau} (\sin\frac{\varphi}{2})^{-\alpha-\frac{3}{2}} (\cos\frac{\varphi}{2})^{-\beta-\frac{1}{2}} \right) + O\left(n^{\alpha-\frac{1}{2}} (\sin\frac{\varphi}{2})^{-\alpha-\frac{5}{2}} (\cos\frac{\varphi}{2})^{-\beta-\frac{3}{2}} \right) & \text{for } \frac{1}{n} \le \varphi < \pi - \frac{1}{n} \quad (4.7.3). \end{cases}$$

Proof: For $0 \le \varphi < \frac{1}{n}$

$$M_n(\varphi) = O(1) \sum_{k=0}^{n} a_{n,k}\, k^{2\alpha+2} \quad \text{by } (4.1.1)$$

$$= O(n^{2\alpha+2}) \sum_{k=0}^{n} a_{n,k}$$

$$= O(n^{2\alpha+2}) A_{n,n}$$

$$= O(n^{2\alpha+2})$$

For $\pi - \frac{1}{n} \le \varphi \le \pi$

$$M_n(\varphi) = O(1) \sum_{k=0}^{n} a_{n,k}\, k^{\alpha+1} k^{\beta} \quad \text{by } (4.1.2)$$

$$= O(1) \sum_{k=0}^{n} a_{n,k}\, k^{\alpha+\beta+1}$$

$$= O(n^{\alpha+\beta+1}) \sum_{k=0}^{n} a_{n,k}$$

$$= O(n^{\alpha+\beta+1}) A_{n,n}$$

$$= O(n^{\alpha+\beta+1})$$

For $\frac{1}{n} \le \varphi < \pi - \frac{1}{n}$

$$M_n(\varphi) = O(1)\, 2^{\alpha+\beta+1} \sum_{k=0}^{n} \left[a_{n,k} \frac{2^{-(\alpha+\beta+1)}}{\Gamma(\alpha+1)} k^{\alpha+1}\, k^{-\frac{1}{2}} k(\varphi) \left\{ \cos(N\varphi+\nu) + \frac{O(1)}{k\sin\varphi} \right\} \right] \quad \text{by } (4.1.3)$$

$$=O(1)\sum_{k=0}^{n} a_{n,k}\, k^{\alpha+\frac{1}{2}} (\sin\tfrac{\varphi}{2})^{-\alpha-\frac{3}{2}} (\cos\tfrac{\varphi}{2})^{-\beta-\frac{1}{2}} \left[\cos\{(k+\rho)\varphi-\gamma\} + \frac{O(1)}{k\sin\varphi}\right]$$

$$=O\left((\sin\tfrac{\varphi}{2})^{-\alpha-\frac{3}{2}} (\cos\tfrac{\varphi}{2})^{-\beta-\frac{1}{2}} \right)$$

$$\sum_{k=0}^{n} a_{n,k} \cos\{(k+\rho)\varphi-\gamma\}\, k^{\alpha+\frac{1}{2}}$$

$$+O\left(\begin{matrix} (\sin\tfrac{\varphi}{2})^{-\alpha-\frac{5}{2}} \\ (\cos\tfrac{\varphi}{2})^{-\beta-\frac{3}{2}} \end{matrix} \right) \sum_{k=0}^{n} a_{n,k}\, k^{\alpha-\frac{1}{2}} .$$

by Lemma 4.5 & 4.6

$$=O\left(n^{\alpha+\frac{1}{2}} A_{n,\tau} (\sin\tfrac{\varphi}{2})^{-\alpha-\frac{3}{2}} (\cos\tfrac{\varphi}{2})^{-\beta-\frac{1}{2}} \right)$$

$$O\left(n^{\alpha-\frac{1}{2}} (\sin\tfrac{\varphi}{2})^{-\alpha-\frac{5}{2}} (\cos\tfrac{\varphi}{2})^{-\beta-\frac{3}{2}} \right)$$

5. Proof of the Theorem

Following the Obrechkoff [8], the n^{th} partial sum of the Jacobi series (2.1) at the point x =1 is given by

$$S_n(1) = 2^{\alpha+\beta+1}\int_0^{\pi} (\sin\tfrac{\varphi}{2})^{2\alpha+1} (\cos\tfrac{\varphi}{2})^{2\beta+1} f(\cos\varphi)\, S_n^{'}(1,\cos\varphi)\, d\varphi,$$

where $S_n^{'}(1,\cos\varphi)$ denotes the nth partial sum of the series

$$\sum_m \frac{P_m^{(\alpha,\beta)}(1) P_m^{(\alpha,\beta)}(\cos\varphi)}{g_m}$$

where

$$g_m = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)},$$

Rau [12] has shown that

$$S_n^{'}(1,\cos\varphi) = \lambda_n P_n^{(\alpha+1,\beta)}(\cos\varphi).$$

Therefore

$$S_n(1) - A$$

$$=2^{\alpha+\beta+1}\lambda_n \int_0^{n} (\sin\tfrac{\varphi}{2})^{2\alpha+1} (\cos\tfrac{\varphi}{2})^{2\beta+1} (f(\cos\varphi)-A) P_n^{(\alpha+1,\beta)}(\cos\varphi)\, d\varphi$$

$$= 2^{\alpha+\beta+1}\lambda_n \int_0^{\pi} F(\varphi) P_n^{(\alpha+1,\beta)}(\cos\varphi)\, d\varphi,$$

where λ_n is defined as in Lemma 4.7

The matrix mean of the Jacobi series (2.1) at x =1, is given by

$$t_n = \sum_{k=0}^{n} a_{n,k} S_k(1)$$

$$t_n - A = \sum_{k=0}^{n} a_{n,k} \left(S_k(1) - A\right)$$

$$= \int_0^{\pi} F(\varphi) M_n(\varphi)\, d\varphi.$$

In order to prove the theorem, we have to show that

$$I = \int_0^{\pi} F(\varphi) M_n(\varphi)\, d\varphi = o(1) \text{ as n } \to \infty.$$

Let us denotes

$$I = \left[\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\delta} + \int_{\delta}^{\pi-\frac{1}{n}} + \int_{\pi-\frac{1}{n}}^{\pi} \right] F(\varphi) M_n(\varphi)\, d\varphi \qquad (5.1)$$

$$= I_1 + I_2 + I_3 + I_4 \quad \text{say},$$

δ being a suitable constant.

$$I_1 = \int_o^{\frac{1}{n}} F(\varphi) M_n(\varphi)\, d\varphi$$

$$|I_1| = \int_0^{\frac{1}{n}} |F(\varphi)| O(n^{2\alpha+2})\, d\varphi$$

$$= O(n^{2\alpha+2}) \int_0^{\frac{1}{n}} |F(\varphi)|\, d\varphi$$

$$= O(n^{2\alpha+2})\, o\left(\frac{1}{n^{2\alpha+2}\xi(n)\log n} \right), \text{by Lemma 4.3},$$

$$= o(1) \quad \text{as n } \to \infty. \qquad (5.2)$$

In order of to estimate I_2, we employ the asymptotic relation given in 4.7.3), thus

$$I_2 = O\left(\int_{\frac{1}{n}}^{\delta} |F(\varphi)|\, n^{\alpha+\frac{1}{2}} A_{n,\tau} (\sin\tfrac{\varphi}{2})^{-\alpha-\frac{3}{2}} \right)$$

$$+ O\left(\int_{\frac{1}{n}}^{\delta} |F(\varphi)|\, n^{\alpha-\frac{1}{2}} (\sin\tfrac{\varphi}{2})^{-\alpha-\frac{5}{2}} \right) \qquad (5.3)$$

$$= I_{2.1} + I_{2.2}, \quad \text{say}.$$

Now, for $I_{2.1}$, given ϵ> 0 choose δ such that if 0 <t ≤ δ, then

$$\Psi(t) = \int_0^{t} |F(\varphi)\, d\varphi| < \varepsilon \frac{t^{2\alpha+2}}{\xi(\frac{1}{t})\log\frac{1}{t}}.$$

$$I_{2.1} = O\left(n^{\alpha+\frac{1}{2}}\right)\int_{\frac{1}{n}}^{\delta}\frac{|F(\varphi)|A_{n,\tau}}{\varphi^{\alpha+\frac{3}{2}}}d\varphi$$

$$= O\left(n^{\alpha+\frac{1}{2}}\right)\left[\left(\left\{\frac{A_{n,\tau}}{\varphi^{\alpha+\frac{3}{2}}}\Psi(\varphi)\right\}_{\frac{1}{n}}^{\delta}\right) - \left(\int_{\frac{1}{n}}^{\delta}\Psi(\varphi)\, d\left(\frac{A_{n,\tau}}{\varphi^{\alpha+\frac{3}{2}}}\right)\right)\right] \quad (5.4)$$

$= I_{2.1.1} + I_{2.1.2}$

We have, $I_{2.1.1}$

$$I_{2.1.1} = O(n^{\alpha+\frac{1}{2}})\Psi(\delta)\frac{A_{n,\left[\frac{1}{\delta}\right]}}{\delta^{\alpha+\frac{3}{2}}}$$

$$< O(n^{\alpha+\frac{1}{2}})\varepsilon\frac{\delta^{\alpha+\frac{1}{2}}}{\xi(\frac{1}{\delta})\log\frac{1}{\delta}}A_{n,\left[\frac{1}{\delta}\right]} \quad (5.5)$$

$$= O(n^{\alpha+\frac{1}{2}}A_{n,\left[\frac{1}{\delta}\right]})\varepsilon\frac{\delta^{\alpha+\frac{1}{2}}}{\xi(\frac{1}{\delta})\log\frac{1}{\delta}}$$

$= o(1)$ as $n \to \infty$.

Again, for $I_{2.1.2}$

$$I_{2.1.2} = O(n^{\alpha+\frac{1}{2}})\int_{\frac{1}{n}}^{\delta}|\Psi(\varphi)|\left|d\left(\frac{A_{n,\tau}}{\varphi^{\alpha+\frac{3}{2}}}\right)\right|$$

$$< O(n^{\alpha+\frac{1}{2}})\varepsilon\int_{\frac{1}{n}}^{\delta}\frac{\varphi^{2\alpha+2}}{\xi(\frac{1}{\varphi})\log\frac{1}{\varphi}}\left|d\left(\frac{A_{n,\tau}}{\varphi^{\alpha+\frac{3}{2}}}\right)\right|$$

and using the change of variables $x = \frac{1}{\varphi}$, we get (assuming that δ< 1),

$$= O(n^{\alpha+\frac{1}{2}})\varepsilon\int_{\frac{1}{\delta}}^{n}\frac{x^{-2\alpha-2}}{\xi(x)\log x}\left|d\left(x^{\alpha+\frac{3}{2}}A_{n,[x]}\right)\right|$$

$$= O(n^{\alpha+\frac{1}{2}})\varepsilon\left[\int_{\frac{1}{\delta}}^{n}\frac{x^{-\alpha-\frac{3}{2}}}{\xi(x)\log x}A_{n,[x]}\,dx + \int_{\frac{1}{\delta}}^{n}\frac{x^{-\alpha-\frac{1}{2}}}{\xi(x)\log x}\left|d A_{n,[x]}\right|\right] \quad (5.6)$$

$= I_{2.1.2.1} + I_{2.1.2.2}$

If m is the integers with $m \le \frac{1}{\delta} \le m+1$, then

$$I_{2.1.2.1} = O(n^{\alpha+\frac{1}{2}})\varepsilon\int_{\frac{1}{\delta}}^{n}\frac{x^{-\alpha-\frac{3}{2}}}{\xi(x)\log x}A_{n,[x]}\,dx$$

$$= O(n^{\alpha+\frac{1}{2}})\varepsilon\left[\sum_{k=n-m}^{n}a_{n,k}\int_{\frac{1}{\delta}}^{m+1}\frac{x^{-\alpha-\frac{3}{2}}}{\xi(x)\log x}dx + \sum_{k=m+1}^{n-1}A_{n,k}\int_{k}^{k+1}\frac{x^{-\alpha-\frac{3}{2}}}{\xi(x)\log x}dx\right] \quad (5.7)$$

$$= O(n^{\alpha+\frac{1}{2}})\varepsilon\left[\frac{A_{n,m}\,m^{-\alpha-\frac{3}{2}}}{\xi(m)\log m} + \sum_{k=m+1}^{n-1}\frac{A_{n,k}\,k^{-\alpha-\frac{3}{2}}}{\xi(k)\log k}\right]$$

$$= O(n^{\alpha+\frac{1}{2}})\,\varepsilon\,O(n^{-\alpha-\frac{1}{2}}),\ by\,(3.2)$$

$= o(1)$ as $n \to \infty$.

Now, for $I_{2.1.2.2}$,

$$I_{2.1..2..2} = O(n^{\alpha+\frac{1}{2}})\varepsilon\int_{\frac{1}{\delta}}^{n}\frac{x^{-\alpha-\frac{1}{2}}}{\xi(x)\log x}\left|d A_{n,[x]}\right|$$

$$= O(n^{\alpha+\frac{1}{2}})\varepsilon\sum_{k=m+1}^{n}a_{n,n-k}\frac{k^{-\alpha-\frac{1}{2}}}{\xi(k)\log k}$$

$$= O(n^{\alpha+\frac{1}{2}})\varepsilon\left[\sum_{k=m+1}^{n}A_{n,k}\left(\frac{k^{-\alpha-\frac{1}{2}}}{\xi(k)\log k} - \frac{(k+1)^{-\alpha-\frac{1}{2}}}{\xi(k+1)\log(k+1)}\right) + A_{n,n}\frac{(n+1)^{-\alpha-\frac{1}{2}}}{\xi(n+1)\log(n+1)} - A_{n,m}\frac{(m+1)^{-\alpha-\frac{1}{2}}}{\xi(m+1)\log(m+1)}\right]$$

by Abel's Lemma

$= o(1)$ as $n \to \infty$. (5.8)

Collecting (5.3) – (5.8), we get

$I_2 = o(1)$ as $n \to \infty$. (5.9)

Considering I_3, we have

$$I_3 = \int_{\delta}^{\pi-\frac{1}{n}}\frac{|F(\varphi)|A_{n,\tau}n^{\alpha+\frac{1}{2}}}{(\sin\frac{\varphi}{2})^{\alpha+\frac{3}{2}}(\cos\frac{\varphi}{2})^{\beta+\frac{1}{2}}}d\varphi + \int_{\delta}^{\pi-\frac{1}{n}}\frac{|F(\varphi)|n^{\alpha-\frac{1}{2}}}{(\sin\frac{\varphi}{2})^{\alpha+\frac{5}{2}}(\cos\frac{\varphi}{2})^{\beta+\frac{3}{2}}}d\varphi$$

$$= O\left(n^{\alpha+\frac{1}{2}}A_{n,\left[\frac{1}{\delta}\right]}\right)\int_{\delta}^{\pi-\frac{1}{n}}\frac{|f(\cos\varphi)-A|}{\left(\cos\frac{\varphi}{2}\right)^{\beta-\frac{1}{2}}\cos\frac{\varphi}{2}}d\varphi$$

$$+ O(n^{\alpha-\frac{1}{2}})\int_{\delta}^{\pi-\frac{1}{n}}|f(\cos\varphi)-A|\left(\cos\frac{\varphi}{2}\right)^{\beta-\frac{1}{2}}d\varphi$$

$$=O\left(n^{\alpha+\frac{1}{2}}A_{n,\left[\frac{1}{\delta}\right]}\right)+O(n^{\alpha-\frac{1}{2}})$$

$$= o(1)+o(1) \quad \text{as} \quad n\to\infty,$$

by the hypothesis of theorem.

$$= o(1) \quad \text{as } n \to \infty. \qquad (5.10)$$

Finally, we consider I_4,

$$I_4 = O(n^{\alpha+\beta+1})\int_{\pi-\frac{1}{n}}^{\pi} |F(\varphi)|\, d\varphi$$

$$= O(n^{\alpha+\beta+1})\int_{\pi-\frac{1}{n}}^{\pi} \frac{|f(\cos\varphi)-A|}{\left(\sin\frac{\varphi}{2}\right)^{2\alpha+1}\left(\cos\frac{\varphi}{2}\right)^{2\beta+1}}\, d\varphi,$$

taking $\pi-\varphi = t.$

$$= O(n^{\alpha+\beta+1})\int_0^{\frac{1}{n}} |f(-\cos t)-A|\, t^{2\beta+1}\, dt$$

$$= O(n^{\alpha-\frac{1}{2}})\int_0^{\frac{1}{n}} |f(-\cos t)-A|\, t^{\beta-\frac{1}{2}}\, dt$$

$$= o(1) \quad \text{as } n \to \infty. \text{ by(3.3).} \qquad (5.11)$$

Collecting (5.1), (5.2), (5.9), (5.10) and (5.11) we get

$$= o(1) \quad \text{as } n \to \infty.$$

Thus, theorem is completely established.

6. Applications

The following particular cases are obtained:

(1) The result of Gupta [4] becomes particular case of our main theorem if,

$$a_{n,k} = \frac{p_{n-k}}{P_n} \quad where \quad P_n = \sum_{k=0}^{n} p_k \neq 0 \quad and \quad \xi(t) = 1 \forall t.$$

(2) The result of Chaudhary [3] becomes particular case of our theorem if,

$$a_{n,k} = \frac{p_{n-k}}{P_n} \quad and \quad \xi(t) = \frac{P_{[t]}}{t\ p_{[t]} \log t} \forall t.$$

(3) The result of Khare and Tripathi [5] becomes particular case of our main theorem if,

$$a_{n,k} = \frac{p_{n-k}\, q_k}{R_n} \; where \; R_n = \sum_{k=0}^{n} p_k\, q_{n-k} \neq 0 \; and \; \xi(t) = 1 \forall t.$$

7. Conclusion

Cesàro, Nörlund, generalized Nörlund Summability methods are the particular cases of matrix Summablity method. In this paper matrix Summability method taken with a condition (3.1) on the Jacobi series (2.1) so that series (2.1) is summable at x=1 to sum A. The result of Gupta [4], Chaudhary [3] and Khare and Tripathi [5] are particular cases of my result.

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