

# g-reciprocal Continuity in Symmetric Spaces

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**Abstract** In this paper, we obtain a common fixed point theorem by employing the notion of g-reciprocal continuity in symmetric spaces. We demonstrate that g-reciprocal continuity ensures the existence of common fixed point under strict contractive conditions, which otherwise do not ensure the existence of fixed points.

**Keywords:** fixed point theorems, symmetric spaces, g-reciprocal continuity, noncompatible mappings, g-compatible

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## 1. Introduction

Fixed point theory of strict contractive conditions constitutes a very important class of mappings and includes contraction mappings as their subclass. It may be observed that strict contractive conditions do not ensure the existence of common fixed points unless some strong condition is assumed either on the space or on the mappings. In such cases either the space is taken to be compact or some sequence of iterates is assumed to be Cauchy sequence. The study of common fixed points of strict contractive conditions using noncompatibility was initiated by Pant [9]. Motivated by Pant [9] researchers [1,3,6,7,11,12,13] of this domain obtained common fixed point results for strict contractive conditions under generalized metric spaces. The significance of this paper lies in the fact that we can obtain fixed point theorems for g-reciprocally continuous mappings in generalized strict contractive conditions without assuming any strong conditions on the space or on the mappings.

## 2. Preliminaries

**Definition 2.1** [14]. Let  $X$  be a non-empty set. A symmetric on a set  $X$  is a real valued function  $d : X \times X \rightarrow R$  such that,

- i.  $d(x, y) \geq 0, \forall x, y \in X,$
- ii.  $d(x, y) = 0 \Leftrightarrow x = y,$
- iii.  $d(x, y) = d(y, x).$

Let  $d$  be a symmetric on a set  $X$  and for  $\epsilon > 0$  and any  $x \in X$ , let  $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ . A topology  $t(d)$  on  $X$  is given by  $U \in t(d)$  if and only if for each  $x \in U, B(x, \epsilon) \subset U$  for some  $\epsilon > 0$ . A symmetric  $d$  is a semi-metric if for each  $x \in X$  and each  $\epsilon > 0, B(x, \epsilon)$  is a neighborhood of  $x$  in the topology  $t(d)$ . There are several

concepts of completeness in this setting. A sequence is a  $d$ -Cauchy if it satisfies the usual metric condition.

**Definition 2.2.** [14]. Let  $(X, d)$  be a symmetric (semi-metric) space.

(1).  $(X, d)$  is  $S$ -Complete if for every  $d$ -Cauchy sequence  $\{x_n\}$  there exists  $x$  in  $X$  with  $\lim d(x_n, x) = 0$ .

(2).  $(X, d)$  is  $d$ -Cauchy Complete if for every  $d$ -Cauchy sequence  $\{x_n\}$  there exists  $x$  in  $X$  with  $\lim x_n = x$  with respect to  $t(d)$ .

(3).  $S : X \rightarrow X$  is  $d$ -Continuous if  $\lim d(x_n, x) = 0$  implies  $\lim d(Sx_n, Sx) = 0$

(4).  $S : X \rightarrow X$  is  $t(d)$  continuous if  $\lim x_n = x$  with respect to  $t(d)$  implies  $\lim S(x_n) = Sx$  with respect to  $t(d)$ .

The following two axioms were given by Wilson [14].

**Definition 2.3.** Let  $(X, d)$  be a symmetric (semi-metric) space.

$W_1$ : Given  $\{x_n\}, x$  and  $y$  in  $X, d(x_n, x) \rightarrow 0$  and  $d(x_n, y) \rightarrow 0 \Rightarrow x = y$ .

$W_2$ : Given  $\{x_n\}, \{y_n\}$  and an  $x$  in  $X, d(x_n, x) \rightarrow 0$  and  $d(x_n, y_n) \rightarrow 0 \Rightarrow d(y_n, x) \rightarrow 0$ .

**Definition 2.4.** [4]. Two self maps  $f$  and  $g$  of a metric space  $(X, d)$  are called compatible if

$\lim d(fgx_n, gfx_n) = 0$ , whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_n fx_n = \lim_n gx_n = t$  for some  $t$  in  $X$ .

The definition of compatibility implies that the mappings  $f$  and  $g$  will be noncompatible if there exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_n fx_n = \lim_n gx_n = t$  for some  $f$  in  $gfx_n \rightarrow gt$ . but  $ffn$  is either non zero or nonexistent.

**Definition 2.5.** [1]. Two self maps  $f$  and  $g$  of a metric space  $(X, d)$  are said to satisfy property (E. A.) if there

exists a sequence  $\{x_n\}$  in  $X$  such that  $\lim_n fx_n = \lim_n gx_n = t$  for some  $t$  in  $X$ .

**Definition 2.6 [5].** Two self maps  $f$  and  $g$  of a metric space  $(X, d)$  are called  $g$ -compatible if  $\lim_n d(ffx_n, gfx_n) = 0$ , whenever  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_n fx_n = \lim_n gx_n = t$  for some  $t$  in  $X$ .

**Definition 2.7 [8].** Two selfmappings  $f$  and  $g$  of a metric space  $(X, d)$  are called reciprocally continuous iff  $fgx_n \rightarrow ft$  and  $gfx_n \rightarrow gt$  whenever  $\{x_n\}$  is a sequence such that  $\lim_n fx_n = \lim_n gx_n = t$  for some  $t$  in  $X$ .

**Definition 2.8 [10].** Two selfmappings  $f$  and  $g$  of a metric space  $(X, d)$  are called  $g$ -reciprocally continuous iff  $ffx_n \rightarrow ft$  and  $gfx_n \rightarrow gt$  whenever  $\{x_n\}$  is a sequence such that  $\lim_n fx_n = \lim_n gx_n = t$  for some  $t$  in  $X$ .

It may be observed that if  $f$  and  $g$  are both continuous then they are obviously  $g$ -reciprocally continuous but the converse is not true. It may also be observed that  $g$ -reciprocal continuity is independent of the notion of reciprocal continuity. The following examples illustrate this fact.

**Example 2.1. [9].** Let  $X = [2, 20]$  and  $d$  be the symmetric (semi-metric)  $d(x, y) = (x - y)^2$  on  $X$ . Define  $f, g : X \rightarrow X$  as follows

$$f2 = 2 \text{ if } x = 2 \text{ or } x > 5, fx = 4 \text{ if } 2 < x \leq 5,$$

$$g2 = 2, gx = 18 \text{ if } 2 < x \leq 5, gx = \frac{(x+1)}{3} \text{ if } x > 5.$$

Then  $f$  and  $g$  are  $g$ -reciprocally continuous but not reciprocally continuous. To see this let us consider the sequence  $x_n = 5 + \frac{1}{n}$ . Then  $fx_n \rightarrow 2, gx_n \rightarrow 2,$   
 $\lim_{n \rightarrow \infty} fgx_n \neq f2, \lim_{n \rightarrow \infty} gfx_n = 2 = g2$  and  
 $\lim_{n \rightarrow \infty} ffx_n = 2 = f2$ . Thus  $f$  and  $g$  are  $g$ -reciprocally continuous but they are not reciprocally continuous.

**Example 2.2. [9].** Let  $X = [2, 20]$  and  $d$  be symmetric (semi-metric)  $d(x, y) = (x - y)^2$  on  $X$ . Define  $f, g : X \rightarrow X$  as follows

$$f2 = 2, fx = 6 \text{ if } 2 < x \leq 5, fx = \frac{(x+5)}{5} \text{ if } x > 5,$$

$$gx = 2 \text{ if } x = 2 \text{ or } x > 5, gx = \frac{(x+4)}{3} \text{ if } 2 < x \leq 5.$$

Then  $f$  and  $g$  are reciprocally continuous but not  $g$ -reciprocally continuous. To see this let us consider the sequence  $x_n = 5 + \frac{1}{n}$ . Then  $fx_n \rightarrow 2, gx_n \rightarrow 2,$   
 $\lim_{n \rightarrow \infty} fgx_n = 2 = f2, \lim_{n \rightarrow \infty} gfx_n = 2 = g2$  and

$\lim_{n \rightarrow \infty} ffx_n = 6 \neq f2$ . Thus  $f$  and  $g$  are reciprocally continuous but they are not  $g$ -reciprocally continuous.

Examples 2.1 and 2.2 clearly show that reciprocal continuity and  $g$ -reciprocally continuous reciprocal continuity are independent of each other.

### 3. Main Results

**Theorem 3.1.** Let  $f$  and  $g$  be  $g$ -reciprocally continuous self mappings of a symmetric space  $(X, d)$  satisfying

$$(i) d(fx, fy) < \max \left\{ \begin{array}{l} d(gx, gy), \\ \frac{d(fx, gx) + d(fy, gy)}{2}, \\ \frac{d(fx, gy) + d(fy, gx)}{2} \end{array} \right\},$$

whenever the right hand side is nonzero. Suppose  $f$  and  $g$  satisfy property (E. A.). If  $f$  and  $g$  are  $g$ -compatible then  $f$  and  $g$  have a unique common fixed point.

**Proof:** Since  $f$  and  $g$  satisfy property (E. A.), there exists a sequence  $\{x_n\}$  in  $X$  such that  $fx_n \rightarrow t$  and  $gx_n \rightarrow t$  for some  $t$  in  $X$ . Suppose that  $f$  and  $g$  are  $g$ -compatible. Then  $\lim_n d(ffx_n, gfx_n) = 0$ .  $g$ -reciprocally continuity of  $f$  and  $g$  implies that  $ffx_n \rightarrow ft$  and  $gfx_n \rightarrow gt$ . The last two limits together imply  $ft = gt$ . Since  $g$ -compatibility implies commutativity at coincidence points, i.e.,  $fgt = gft$  and, hence  $fft = fgt = gft = ggt$ . If  $ft \neq fft$  then by using (i), we get

$$d(ft, fft) < \max \left\{ \begin{array}{l} d(gt, gft), \\ \frac{d(ft, gt) + d(fft, gft)}{2}, \\ \frac{d(ft, gft) + d(fft, gt)}{2} \end{array} \right\} = d(ft, fft),$$

a contradiction. Hence  $ft = fft = gft$  and  $ft$  is a common fixed point of  $f$  and  $g$ . This completes the proof of the theorem.

The next example illustrates the above theorem.

**Example 3.1.** Let  $X = [2, 20]$  with the symmetric (semi-metric)  $d(x, y) = (x - y)^2$ . Define  $f, g : X \rightarrow X$  as follows

$$fx = 2 \text{ if } x = 2 \text{ or } x > 5, fx = 6 \text{ if } 2 < x \leq 5,$$

$$g2 = 2, gx = 12 \text{ if } 2 < x \leq 5, gx = (x+1)/3 \text{ if } x > 5.$$

Then  $f$  and  $g$  satisfy all the conditions of Theorem 3.1 and have a unique common fixed point at  $x = 2$ . It can be verified in this example that  $f$  and  $g$  satisfy the contraction condition (i). Furthermore,  $f$  and  $g$  are  $g$ -reciprocally continuous  $g$ -compatible mappings. It is also

obvious that  $f$  and  $g$  are not reciprocally continuous. Here  $f$  and  $g$  are not reciprocally continuous mappings.

**Remark:** In the above result we have not assumed strong conditions, e.g., completeness of the space, containment of the ranges of the mappings, closedness of the range of any one of the involved mappings and continuity of any mapping. We also proved a result using generalized strict contractive condition. It may be observed that strict contractive conditions do not ensure the existence of common fixed points unless the space is assumed compact or the strict contractive condition is replaced by some strong conditions, e.g., a Banach type contractive condition or a  $\phi$ -contractive condition or a Meir-Keeler type contractive condition. In the result established in this paper, we have not assumed any mapping to be continuous. Thus we provide more answers to the problem posed by Rhoades [2] regarding existence a contractive definition which is strong enough to generate a fixed point, but which does not force the map to be continuous at the fixed point.

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