American Journal of Mathematical Analysis, 2013, Vol. 1, No. 3, 33-38
Available online at http://pubs.sciepub.com/ajma/1/3/2
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DOI:10.12691/ajma-1-3-2

Hermite-Hadamard's Inequalities for Preinvex Function via Fractional Integrals and Related Fractional Inequalities

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Received August 12, 2013; Revised August 24, 2013; Accepted August 28, 2013

Abstract In this paper, the author has established Hermite- Hadamard's inequalities for preinvex functions and has extended some estimates of the right side of a Hermite- Hadamard type inequalities for preinvex functions via fractional integrals.

Keywords: *Hermite-Hadamard's inequalities, invex set, preinvex function, fractional integrals*

Cite This Article: İmdat İşcan, "Hermite-Hadamard's Inequalities for Preinvex Function via Fractional Integrals and Related Fractional Inequalities." *American Journal of Mathematical Analysis* 1, no. 3 (2013): 33-38. doi: 10.12691/ajma-1-3-2.

1. Introduction

Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \qquad (1)$$

This doubly inequality is known in the literature as Hermite-Hadamard integral inequality for convex mapping.We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. For several recent results concerning the inequality (1) we refer the interested reader to [3,5,6,8,9,11,18,21,22] and the references cited therein.

Definition 1.1 *The function* $f:[a,b] \subset \mathbb{R} \to \mathbb{R}$ *is said to be convex if the following inequality holds:*

$$f\left(tx+(1-t)y\right) \le tf(x)+(1-t)f(y)$$

for all $x, y \in [a,b]$ and $t \in [0,1]$. We say that f is concave if $(-f)$ is convex.

In [18] Pearce and Pečarić established the following result connected with the right part of (1).

Theorem 1.2 *Let* $f: I^\circ \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $a, b \in I^\circ$ *with* $a < b$, *and let* $q \ge 1$. *If the mapping* $|f'|^q$ *convex on* $[a,b]$, *then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q + |f'(b)|^q}{2}\right]^{\frac{1}{q}} \qquad (2)$$

The classical Hermite- Hadamard inequality provides estimates of the mean value of a continuous convex function $f:[a,b] \to \mathbb{R}$.

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.3 *Let* $f \in L[a,b]$. *The Riemann-Liouville integrals* $J_{a^+}^{\alpha} f$ *and* $J_{b^-}^{\alpha} f$ *of oder* $\alpha > 0$ *with* $a \ge 0$ *are defined by*

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \; x > a$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \; x < b$$

respectively, where $\Gamma(\alpha)$ *is the Gamma function and* $J_{a^+}^{0} f(x) = J_{b^-}^{0} f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral. Properties concerning this operator can be found ([7,12,17]).

For some recent result connected with fractional integral see ([4,19,20,22]).

In [19] Sarıkaya et al. proved the following Hadamard type inequalities for fractional integrals as follows.

Theorem 1.4 *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\le a<b$ and $f\in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(x)+J_{b^-}^{\alpha}f(x)\right]\le\frac{f(a)+f(b)}{2} \quad (3)$$

with $\alpha>0$.

Using the following identity Sarıkaya et al. in [17] established the following result which hold for differentiable functions.

Lemma 1.5 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(x)+J_{b^-}^{\alpha}f(x)\right]\\&=\frac{b-a}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'(ta+(1-t)b)\,dt.\end{aligned} \quad (4)$$

Theorem 1.6 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is a convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(x)+J_{b^-}^{\alpha}f(x)\right]\right|\\&\le\frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(a)|+|f'(b)|\right]\end{aligned} \quad (5)$$

In recent years several extentions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson in [8]. Weir and Mond [23] introduced the concept of preinvex functions and applied it to the establisment of the sufficient optimality conditions and duality in nonlinear programming. Pini [24] introduced the concept of prequasiinvex as a generalization of invex functions. Later, Mohan and Neogy [24] obtained some properties of generalized preinvex functions. Noor [13,14,15,16] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In recent papers Barani et al. in [2] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

In this paper we generalized the results in [2] and [19] for preinvex functions via fractional integrals. Now we recall some notions in invexity analysis which will be used throught the paper (see [1,23] and references therein)

Let $f:A\to\mathbb{R}$ and $\eta:A\times A\to\mathbb{R}^n$, where A is a nonempty set in $\mathbb{R}^n$, be continuous functions.

Definition 1.7 *The set $A\subseteq\mathbb{R}^n$ is said to be invex with respect to $\eta(.,.)$, if for every $x,y\in A$ and $t\in[0,1]$, $x+t\eta(y,x)\in A$.*

The invex set A is also called a $\eta-$connected set. It is obvious that every convex set is invex with respect to $\eta(y,x)=y-x$, but there exist invex sets which are not convex [1].

Definition 1.8 *The function f on the invex set A is said to be preinvex with respect to η if*

$$f(x+t\eta(y,x))\le(1-t)f(x)+tf(y),\ \forall x,y\in A,\ t\in[0,1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

Mohan and Neogy [24] introduced condition C defined as follows

Condition C: Let $A\subseteq\mathbb{R}^n$ be an invex subset with respect to $\eta:A\times A\to\mathbb{R}^n$. We say that the function η satisfies the condition C if for any $x,y\in A$ and any $t\in[0,1]$,

$$\begin{aligned}\eta(y,y+t\eta(x,y))&=-t\eta(x,y)\\\eta(x,y+t\eta(x,y))&=(1-t)\eta(x,y).\end{aligned} \quad (6)$$

Note that for every $x,y\in A$ and every $t_1,t_2\in[0,1]$ from condition C, we have

$$\eta(y+t_2\eta(x,y),y+t_1\eta(x,y))=(t_2-t_1)\eta(x,y).$$

we will use the condition in our main results.

In [16], Noor proved the Hermite-Hadamard inequality for the preinvex functions as follows:

Theorem 1.9 *Let $f:K=[a,a+\eta(b,a)]\to(0,\infty)$ be a preinvex function on the interval of real numbers K^o (the interior of K) and $a,b\in K^o$ with $a<a+\eta(b,a)$. Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right)\le\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\le\frac{f(a)+f(b)}{2} \quad (7)$$

In [2] Barani, Gahazanfari, and Dragomir proved the following theorems:

Theorem 1.10 *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on A then, for every $a,b\in A$ with $\eta(b,a)\ne0$ the following inequalities holds*

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right|\\&\le\frac{|\eta(b,a)|}{8}\left[|f'(a)|+|f'(b)|\right].\end{aligned} \quad (8)$$

Theorem 1.11 *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. Assume that $p\in\mathbb{R}$ with $p>1$. If $|f'|^{\frac{p}{p-1}}$ is preinvex on A then, for every $a,b\in A$ with $\eta(b,a)\ne0$ the following inequalities holds*

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)dx\right|$$

$$\leq\frac{|\eta(b,a)|}{2(p+1)^{\frac{1}{p}}}\left[|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}. \tag{9}$$

2. Main Results

Throughout this section, Let $A\subseteq\mathbb{R}$ be an open invex subset. In this section, firstly we will establish Hermite-Hadamard's inequalities for preinvex functions via fractional integrals. Secondly we will introduce some generalizations of the right side of a Hermite- Hadamard type inequalities for functions whose first derivatives absolute values are preinvex via fractional integrals.

Theorem 2.1 *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$. If $f:[a,a+\eta(b,a)]\to(0,\infty)$ is a preinvex function, $f\in L[a,a+\eta(b,a)]$ and η satisfies condition C then, the following inequalities for fractional integrals holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right)$$
$$\leq\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right] \tag{10}$$
$$\leq\frac{f(a)+f(a+\eta(b,a))}{2}\leq\frac{f(a)+f(b)}{2}$$

with $\alpha>0$.

Proof. Since $a,b\in A$ and A is an invex set with respect to η, for every $t\in[0,1]$, we have $a+t\eta(b,a)\in A$. By preinvexity of f, we have for every $x,y\in[a,a+\eta(b,a)]$ with $t=\frac{1}{2}$

$$f\left(x+\frac{\eta(y,x)}{2}\right)\leq\frac{f(x)+f(y)}{2}$$

i.e. with $x=a+(1-t)\eta(b,a)$, $y=a+t\eta(b,a)$ from equality (6) we get

$$2f\left(a+(1-t)\eta(b,a)+\frac{\eta(a+t\eta(b,a),a+(1-t)\eta(b,a))}{2}\right)$$
$$=2f\left(a+(1-t)\eta(b,a)+\frac{(2t-1)\eta(b,a))}{2}\right) \tag{11}$$
$$=2f\left(\frac{2a+\eta(b,a)}{2}\right)$$
$$\leq f\left(a+(1-t)\eta(b,a)\right)+f\left(a+t\eta(b,a)\right)$$

Multiplying both sides (11) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\frac{2}{\alpha}f\left(\frac{2a+\eta(b,a)}{2}\right)$$
$$\leq\int_0^1 t^{\alpha-1}f\left(a+(1-t)\eta(b,a)\right)dt+\int_0^1 t^{\alpha-1}f\left(a+t\eta(b,a)\right)dt$$
$$=\frac{1}{\eta^{\alpha}(b,a)}\left[\int_a^{a+\eta(b,a)}\left(a+\eta(b,a)-u\right)^{\alpha-1}f(u)du\right.$$
$$\left.+\int_a^{a+\eta(b,a)}\left(u-a\right)^{\alpha-1}f(u)du\right]$$
$$=\frac{\Gamma(\alpha)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]$$

i.e.

$$f\left(\frac{2a+\eta(b,a)}{2}\right)$$
$$\leq\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{a+\eta(b,a)^-}f(a)\right]$$

and the fist inequality is proved.

For the proof of the second inequality in (11) we first note that if f is a preinvex function on $[a,a+\eta(b,a)]$ and the mapping η satisfies condition C then for every $t\in[0,1]$, from inequality (6) it yields

$$f\left(a+t\eta(b,a)\right)=f\left(a+\eta(b,a)+(1-t)\eta(a,a+\eta(b,a))\right)$$
$$\leq tf\left(a+\eta(b,a)\right)+(1-t)f(a) \tag{12}$$

and similarly

$$f\left(a+(1-t)\eta(b,a)\right)=f\left(a+\eta(b,a)+t\eta(a,a+\eta(b,a))\right)$$
$$\leq(1-t)f\left(a+\eta(b,a)\right)+tf(a).$$

By adding these inequalities we have

$$f\left(a+t\eta(b,a)\right)+f\left(a+(1-t)\eta(b,a)\right)\leq f(a)+f\left(a+\eta(b,a)\right) \tag{13}$$

Then multiplying both (13) by $t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\int_0^1 t^{\alpha-1}f\left(a+t\eta(b,a)\right)dt+\int_0^1 t^{\alpha-1}f\left(a+(1-t)\eta(b,a)\right)dt$$
$$\leq\left[f(a)+f\left(a+\eta(b,a)\right)\right]\int_0^1 t^{\alpha-1}dt.$$

i.e.

$$\frac{\Gamma(\alpha)}{\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]$$
$$\leq\frac{f(a)+f\left(a+\eta(b,a)\right)}{\alpha}.$$

Using the mapping η satisfies condition C the proof is completed.

Remark 2.2 a) *If in Theorem 2.1, we let $\eta(b,a)=b-a$, then inequality (10) become inequality (3) of Theorem 1.4.*

b) If in Theorem 2.1, we let $\alpha = 1$, then inequality (10) become inequality (7) of Theorem 1.9.

Now we give the following lemma which is a generalization of Lemma 1.5 to invex setting.

Lemma 2.3 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b,a)$. If $f : A \to \mathbb{R}$ is a differentiable function such that $f' \in L[a, a+\eta(b,a)]$ then, the following equality holds:*

$$\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+} f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a)\right] = \frac{\eta(b,a)}{2}\int_0^1 \left[t^{\alpha} - (1-t)^{\alpha}\right] f'(a+t\eta(b,a))\,dt \quad (14)$$

Proof. It suffices to note that

$$I = \int_0^1 \left[t^{\alpha} - (1-t)^{\alpha}\right] f'(a+t\eta(b,a))\,dt = \left[\int_0^1 t^{\alpha} f'(a+t\eta(b,a))\,dt\right] + \left[-\int_0^1 (1-t)^{\alpha} f'(a+t\eta(b,a))\,dt\right] = I_1 + I_2 \quad (15)$$

integrating by parts

$$I_1 = \int_0^1 t^{\alpha} f'(a+t\eta(b,a))\,dt = t^{\alpha}\frac{f(a+t\eta(b,a))}{\eta(b,a)}\Bigg|_0^1 - \int_0^1 \alpha t^{\alpha-1}\frac{f(a+t\eta(b,a))}{\eta(b,a)}\,dt \quad (16)$$

$$= \frac{f(a+\eta(b,a))}{\eta(b,a)} - \frac{\alpha}{\eta(b,a)}\int_a^{a+\eta(b,a)}\left(\frac{x-a}{\eta(b,a)}\right)^{\alpha-1}\frac{f(x)}{\eta(b,a)}\,dx = \frac{f(a+\eta(b,a))}{\eta(b,a)} - \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,a)} J^{\alpha}_{(a+\eta(b,a))^-} f(a)$$

and similarly we get,

$$I_2 = -\int_0^1 (1-t)^{\alpha} f'(a+t\eta(b,a))\,dt = -(1-t)^{\alpha}\frac{f(a+t\eta(b,a))}{\eta(b,a)}\Bigg|_0^1 - \int_0^1 \alpha(1-t)^{\alpha-1}\frac{f(a+t\eta(b,a))}{\eta(b,a)}\,dt \quad (17)$$

$$= \frac{f(a)}{\eta(b,a)} - \frac{\alpha}{\eta(b,a)}\int_a^{a+\eta(b,a)}\left(\frac{a+t\eta(b,a)-x}{\eta(b,a)}\right)^{\alpha-1}\frac{f(x)}{\eta(b,a)}\,dx = \frac{f(a)}{\eta(b,a)} - \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,a)} J^{\alpha}_{a^+} f(a+\eta(b,a))$$

Using (16) and (17) in (15), it follows that

$$I = \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+} f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a)\right].$$

Thus, by multiplying both sides by $\frac{\eta(b,a)}{2}$, we have conclusion (14).

Remark 2.4 *If in Lemma 2.3, we let $\eta(b,a) = b - a$, then equality (14) become inequality (4) of Lemma 1.5.*

Theorem 2.5 *Let $A \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $a < a + \eta(b,a)$. Suppose that $f : A \to \mathbb{R}$ is a differentiable function such that $f' \in L[a, a+\eta(b,a)]$. If $|f'|$ is preinvex function on $[a, a+\eta(b,a)]$ then the following inequality for fractional integrals with $\alpha > 0$ holds:*

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+} f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a)\right]\right| \le \frac{\eta(b,a)}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left[|f'(a)| + |f'(b)|\right] \quad (18)$$

Proof. Using lemma 2.3 and the preinvexity of $|f'|$ we get

$$\left|\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+} f(a+\eta(b,a)) + J^{\alpha}_{(a+\eta(b,a))^-} f(a)\right]\right|$$

$$\le \frac{\eta(b,a)}{2}\int_0^1 \left|t^{\alpha} - (1-t)^{\alpha}\right| \left|f'(a+t\eta(b,a))\right| dt$$

$$\le \frac{\eta(b,a)}{2}\int_0^1 \left|t^{\alpha} - (1-t)^{\alpha}\right| \left[(1-t)|f'(a)| + t|f'(b)|\right] dt$$

$$\le \frac{\eta(b,a)}{2}\left\{\int_0^{1/2}\left[(1-t)^{\alpha} - t^{\alpha}\right]\left[(1-t)|f'(a)| + t|f'(b)|\right] dt + \int_{1/2}^1 \left[t^{\alpha} - (1-t)^{\alpha}\right]\left[(1-t)|f'(a)| + t|f'(b)|\right] dt\right\}$$

$$= \frac{\eta(b,a)}{2}\left[|f'(a)| + |f'(b)|\right]\left(\int_0^{1/2}\left[(1-t)^{\alpha} - t^{\alpha}\right] dt\right)$$

$$= \frac{\eta(b,a)}{2(\alpha+1)}\left(1 - \frac{1}{2^{\alpha}}\right)\left[|f'(a)| + |f'(b)|\right],$$

which completes the proof.

Remark 2.6 *a) If in Theorem 2.5, we let $\eta(b,a) = b - a$, then inequality (18) become inequality (5) of Theorem 1.6.*

b) If in Theorem 2.5, we let $\alpha = 1$, then inequality (18) become inequality (8) of Theorem 1.10.

c) In Theorem 2.5, assume that η satisfies condition C and using inequality (12) for $|f'|$ we get

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right|$$
$$\leq\frac{\eta(b,a)}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\left|f'(a)\right|+\left|f'(a+\eta(b,a))\right|\right].$$

Theorem 2.7 *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$ such that $f'\in L\left[a,a+\eta(b,a)\right]$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function. If $\left|f'\right|^q$ is preinvex function on $\left[a,a+\eta(b,a)\right]$ for some fixed $q>1$ then the following inequality holds:*

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right| \quad (19)$$
$$\leq\frac{\eta(b,a)}{2(\alpha p+1)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha\in\left[0,1\right]$.

Proof. From lemma2.3 and using Hölder inequality with properties of modulus, we have

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right|$$
$$\leq\frac{\eta(b,a)}{2}\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|dt$$
$$\leq\frac{\eta(b,a)}{2}\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}.$$

We know that for $\alpha\in\left[0,1\right]$ and $\forall t_1,t_2\in\left[0,1\right]$,

$$\left|t_1^{\alpha}-t_2^{\alpha}\right|\leq\left|t_1-t_2\right|^{\alpha},$$

therefore

$$\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|^p dt\leq\int_0^1\left|1-2t\right|^{\alpha p}dt$$
$$=\int_0^{1/2}\left[1-2t\right]^{\alpha p}dt+\int_{1/2}^1\left[2t-1\right]^{\alpha p}dt=\frac{1}{\alpha p+1}.$$

Since $\left|f'\right|^q$ is preinvex on $\left[a,a+\eta(b,a)\right]$, we have inequality (19), which completes the proof.

Remark 2.8 *a) If in Theorem 2.7, we let $\eta(b,a)=b-a$ and $\alpha=1$ then inequality (19) become inequality (9) of Theorem1.11.*

b) In Theorem 2.7, assume that η satisfies condition C and using inequality (12) we get

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right|$$
$$\leq\frac{\eta(b,a)}{2(\alpha p+1)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^q+\left|f'(a+\eta(b,a))\right|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 2.9 *Let $A\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:A\times A\to\mathbb{R}$ and $a,b\in A$ with $a<a+\eta(b,a)$. Suppose that $f:A\to\mathbb{R}$ is a differentiable function such that $f'\in L\left[a,a+\eta(b,a)\right]$. If $\left|f'\right|^q$ is preinvex function on $\left[a,a+\eta(b,a)\right]$ for some fixed $q>1$ then the following inequality holds:*

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right| \quad (20)$$
$$\leq\frac{\eta(b,a)}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>0$.

Proof. From lemma2.3 and using Hölder inequality with properties of modulus, we have

$$\left|\frac{f(a)+f\left(a+\eta(b,a)\right)}{2}-\frac{\Gamma(\alpha+1)}{2\eta^{\alpha}(b,a)}\left[J^{\alpha}_{a^+}f(a+\eta(b,a))+J^{\alpha}_{(a+\eta(b,a))^-}f(a)\right]\right|$$
$$\leq\frac{\eta(b,a)}{2}\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|^{\frac{1}{p}+\frac{1}{q}}\left|f'\left(a+t\eta(b,a)\right)\right|dt$$
$$\leq\frac{\eta(b,a)}{2}\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|dt\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left|t^{\alpha}-(1-t)^{\alpha}\right|\left|f'\left(a+t\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}$$

On the other hand, we have

$$\int_0^1 \left| t^\alpha - (1-t)^\alpha \right| dt = \int_0^{1/2} \left[(1-t)^\alpha - t^\alpha \right] dt + \int_{1/2}^1 \left[t^\alpha - (1-t)^\alpha \right] dt$$
$$= \frac{2}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right).$$

Since $|f'|^q$ is preinvex function on A, we obtain

$$\left| f'(a + t\eta(b,a)) \right|^q \le (1-t)\left| f'(a) \right|^q + t\left| f'(b) \right|^q, \ t \in [0,1]$$

and

$$\int_0^1 \left| t^\alpha - (1-t)^\alpha \right| \left| f'(a + t\eta(b,a)) \right|^q dt$$
$$\le \int_0^1 \left| t^\alpha - (1-t)^\alpha \right| \left[(1-t)\left| f'(a) \right|^q + t\left| f'(b) \right|^q \right] dt$$
$$= \int_0^{1/2} \left[(1-t)^\alpha - t^\alpha \right] \left[(1-t)\left| f'(a) \right|^q + t\left| f'(b) \right|^q \right] dt$$
$$+ \int_{1/2}^1 \left[t^\alpha - (1-t)^\alpha \right] \left[(1-t)\left| f'(a) \right|^q + t\left| f'(b) \right|^q \right] dt$$
$$= \frac{1}{\alpha+1}\left(1 - \frac{1}{2^\alpha}\right)\left[\left| f'(a) \right|^q + \left| f'(b) \right|^q \right]$$

from here we obtain inequality (20) which completes the proof.

Remark 2.10 *a) If in Theorem2.9, we let* $\eta(b,a) = b - a$ *and* $\alpha = 1$ *then inequality (20)become inequality (2) Theorem1.2.*

b) In Theorem2.9, assume that η *satisfies condition C, using inequality (12) we get*

$$\left| \frac{f(a) + f(a + \eta(b,a))}{2} - \frac{\Gamma(\alpha+1)}{2\eta^\alpha(b,a)}\left[J_{a^+}^\alpha f(a + \eta(b,a)) + J_{(a+\eta(b,a))^-}^\alpha f(a) \right] \right|$$
$$\le \frac{\eta(b,a)}{(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\left[\frac{\left| f'(a) \right|^q + \left| f'(a + \eta(b,a)) \right|^q}{2} \right]^{\frac{1}{q}}.$$

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