

Common Fixed Points in Cone Metric Spaces

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Abstract In this paper, we prove a unique common fixed point theorem in cone metric spaces without appealing to commutativity conditions. These results generalize several well-known comparable results in the literature.

Keywords: common fixed point, cone metric space, coincidence points

1. Introduction

The study of common fixed points of mappings satisfying certain contractive conditions has been at the centre of vigorous research activity, being the applications of fixed point very important in several areas of Mathematics. In 2007 Huang and Zhang [5] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently, Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [5,14] and the references mentioned therein). Isak Altun, G.Durmaz,[3] have proved some fixed point theorems on ordered cone metric spaces and M.Abbas , B.E.Rhoades [2] have proved common fixed point theorems for mappings without appealing to commutativity conditions in cone metric spaces. Recently, Abbas and Jungck [1] have obtained coincidence point result for two mappings in cone metric spaces. In this paper we prove coincidence point results in cone metric spaces without appealing to commutativity conditions.

In all that follows B is a real Banach Space, and θ denotes the zero element of B . For the mapping $f, g: X \rightarrow X$, let $C(f, g)$ denote the set of coincidence points of f and g , that is $C(f, g) = \{z \in X : fz = gz\}$.

The following definitions are due to Huang and Zhang [5].

Definition 1.1. Let B be a real Banach Space and P a subset of B . The set P is called a cone if and only if:

- P is closed, non –empty and $p \neq \{\theta\}$;
- $a, b \in P, a, b \geq 0, x, y \in P$ implies $ax+by \in P$;
- $x \in P$ and $-x \in P$ implies $x = \theta$.

Definition 1.2. Let P be a cone in a Banach Space B , define partial ordering ' \leq ' with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$ while $X \ll y$ will stand for $y-x \in \text{Int } P$, where $\text{Int } P$ denotes the interior of the set P . This Cone P is called an order cone.

Definition 1.3. Let B be a Banach Space and $P \subset B$ be an order cone. The order cone P is called normal if there exists $L > 0$ such that for all $x, y \in B$,

$$\theta \leq x \leq y \text{ implies } \|x\| \leq L \|y\|$$

The least positive number L satisfying the above inequality is called the normal constant of P .

Definition 1.4. Let X be a nonempty set of B . Suppose that the map $d: X \times X \rightarrow B$ satisfies:

- $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x=y$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Example 1.5. ([5]). Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$ and $d: X \times X \rightarrow E$ such that $d(x, y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.6. Let (X, d) be a cone metric space. We say that $\{x_n\}$ is

- a Cauchy sequence if for every c in B with $c \gg \theta$, there is N such that for all $n, m > N, d(x_n, x_m) \ll c$;
- a convergent sequence if for any $c \gg \theta$, there is an N such that for all $n > N, d(x_n, x) \ll c$, for some fixed x in X . We denote this $x_n \rightarrow x$ (as $n \rightarrow \infty$).

Lemma 1.7. Let (X, d) be a cone metric space, and let P be a normal cone with normal constant K . Let $\{x_n\}$ be a sequence in X . Then

- $\{x_n\}$ converges to x if and only if $d(x_n, x) \rightarrow 0$ ($n \rightarrow \infty$).
- $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \rightarrow 0$ (as $n, m \rightarrow \infty$).

2. Common Fixed Point Theorem

In this section we obtained a common fixed point theorem for mappings without appealing to commutativity conditions, defined on a cone metric space. Which generalizes the results of Abbas and Jungck [1].

The following theorem generalizes the Theorem 2.4 [1].

Theorem 2.1. Let (X, d) be a complete cone metric space and P a normal cone with normal constant K . Suppose that the mappings $f, g: X \rightarrow X$ are such that for some constant $\lambda \in (0,1)$ and for every $x, y \in X$ are two self-maps of X satisfying

$$d(fx, fy) \leq \lambda(d(fx, gy) + d(fy, gx)) \quad (1)$$

If the range of g contains the range of f and $g(X)$ is a complete subspace of X , then f and g have coincidence point. Then, f and g have a unique common fixed point in X .

Proof: Let x_0 be an arbitrary point in X , and let $x_1 \in X$ be chosen such that

$y_0 = f(x_0) = g(x_1)$. Since $f(X) \subseteq g(X)$. Let $x_2 \in X$ be chosen such that

$y_1 = f(x_1) = g(x_2)$. Continuing this process, having chosen $x_n \in X$, we chose $x_{n+1} \in X$

such that $y_n = f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} d(y_n, y_{n-1}) &= d(fx_n, fx_{n-1}) \\ &\leq k(d(fx_n, gx_{n-1}) + d(fx_{n-1}, gx_n)) \\ &\hspace{10em} \text{(by(1))} \\ &\leq k(d(gx_{n+1}, gx_n) + d(gx_n, gx_{n-1})) \quad (2) \\ &\leq k(d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})), \\ &\hspace{10em} \text{for } n=2,3,\dots \end{aligned}$$

$$\begin{aligned} d(y_n, y_{n-1}) - k(d(y_n, y_{n-1})) &\leq d(y_{n-1}, y_{n-2}) \\ \Rightarrow d(y_n, y_{n-1}) &\leq \frac{1}{1-k} d(y_{n-1}, y_{n-2}) \\ &\leq h d(y_{n-1}, y_{n-2}), \text{ where, } h = \frac{1}{1-k} \end{aligned}$$

Now by (2)

$$d(y_n, y_{n-1}) \leq kd(y_{n-1}, y_{n-2}) \leq \dots \leq k^{n-1}d(y_1, y_0)$$

Now we shall show that $\{y_n\}$ is a Cauchy sequence.

By the triangle inequality, for $n > m$, we have

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n-1}) \\ &\quad + d(y_{n-1}, y_{n-2}) + \dots + d(y_{m+1}, y_m) \end{aligned}$$

Hence, as p is a normal cone,

$$d(y_n, y_{n-1}) \leq kd(y_{n-1}, y_{n-2}) \leq \dots \leq k^{n-1}d(y_1, y_0)$$

$$\begin{aligned} \|d(y_n, y_m)\| &\leq L(\|d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \dots \\ &\quad \dots + d(y_{m+1}, y_m)\|), \end{aligned}$$

$$\begin{aligned} &\leq L(\|d(y_n, y_{n-1})\| + \|d(y_{n-1}, y_{n-2})\| + \dots \\ &\quad \dots + \|d(y_{m+1}, y_m)\|) \end{aligned}$$

$$\|d(y_n, y_m)\| \leq L(K^{n-1} + K^{n-2} + \dots + K^m)\|d(y_1, y_0)\|$$

$$\leq \frac{L \cdot K^m}{1-K} \|d(y_1, y_0)\| \rightarrow 0, \text{ as } m \rightarrow \infty$$

From ([5], Lemma 4) it follows that $\{y_n\}$ is a Cauchy sequence. Since $g(X)$ is complete, there exists a q in $g(X)$ such that $y_n \rightarrow q$ as $n \rightarrow \infty$. Consequently, we can find p in X such that $g(p) = q$. We shall show that $f(p) = q$.

From (1)

$$\begin{aligned} d(gx_n, fp) &= d(fx_{n-1}, fp) \leq k(d(fx_{n-1}, gp) + d(fp, gx_{n-1})), \\ &\Rightarrow d(gp, fp) \leq k(d(gp, gp) + d(fp, gp)) \\ &\leq kd(fp, gp) \end{aligned}$$

It follows that, $d(gp, fp) = 0$

Hence,

$$gp = q = fp, \text{ } p \text{ is a coincidence point of } f \text{ and } g \quad (3)$$

Now using (1),

$$d(p, gp) \leq d(p, y_n) + d(y_n, gp)$$

(by the triangle inequality)

$$\begin{aligned} &= d(p, y_n) + d(fx_n, fp) \rightarrow (\text{since } fp = gp) \\ &\leq d(p, y_n) + k(d(fx_n, gp) + d(fp, gx_n)) \end{aligned}$$

From (1.3),

$$\|d(p, gp)\| \leq L(\|d(p, y_n) + k(d(fx_n, gp) + d(fp, gx_n))\|),$$

$$\leq L(\|d(p, y_n)\| + k(\|d(y_n, gp)\|) + \|d(fp, y_{n-1})\|) \text{ as } n \rightarrow \infty$$

$$\leq L(\|d(p, q)\| + k(\|d(q, gp)\| + \|d(fp, q)\|)),$$

$$\leq L(\|d(p, gp)\| + k(\|d(gp, gp)\| + \|d(fp, fp)\|))$$

$$\leq L\|d(p, gp)\| + 0$$

$$\Rightarrow \|d(p, gp)\| = 0$$

That is, $p = gp$

Now,

$$d(fp, p) = d(fp, gp)$$

$$\leq k(d(fp, gp) + d(fp, gp)) \quad \text{(by(1))}$$

$$\leq 0 \quad (\text{since, } fp = gp)$$

$$\Rightarrow \|d(fp, p)\| = 0$$

That is, $fp = p$

Since, $fp = gp$

Therefore, $fp = gp = p$, f and g have a common fixed point.

Uniqueness, let p_1 be another common fixed point of f and g , then

$$\begin{aligned} d(p, p_1) &= d(fp, gp_1) \\ &= d(fp, fp_1) \\ &\leq k(d(fp, gp_1) + d(fp_1, gp)) \text{ (by (1))} \\ &\leq k(d(p, p_1) + d(p_1, p)) \\ &\leq 2k \cdot d(p_1, p) \end{aligned}$$

It follows that, $d(p, p_1) = 0$ that is, $p = p_1$

Therefore, f and g have a unique common fixed point.

This completes the proof.

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