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Numerical Solution of Fractional Bioheat Equation with Constant and Sinusoidal Heat Flux Condition on Skin Tissue

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Abstract Heat transfer in skin tissue is an area of interest for medical sciences. In this paper we intend to study fractional bioheat equation for heat transfer in skin tissue with constant and sinusoidal heat flux condition on skin surface. Numerical solutions are obtained by implicit finite difference method. We study the effect of anomalous diffusion in skin tissue and compare it with normal diffusion, with constant and sinusoidal heat flux. This study intends to find the temperature profiles for different order fractional bioheat equations.

Keywords: *fractional bioheat equation, finite difference method*

1. Introduction

Heat transfer in biological tissue, is usually expressed as bioheat equation. It involves thermal conduction, convection, perfusion of blood, and metabolic heat generation in tissue. Pennes [6] bioheat model is widely used to study the heat transfer in skin tissue, due to its simplicity. In human body skin is the largest living organ. Temperature distribution in skin tissue is very important for medical application like skin cancer, skin burns etc.

Recently fractional order equations have drawn the attention of many researchers and have been focused for many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering etc.

Fractals and fractional calculus have been used to improve the modelling accuracy of many phenomena in natural science. The most important advantage of using fractional differential equations in this and other applications is their non-local property. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. These are more realistic and also easy to make the fractional calculus popular [7]. Numerical solution of fractional diffusion equation by finite difference method have been studied by many researchers, Meerschaert et al.[4], gave a second order accurate numerical approximation for the fractional diffusion equation, Murio [5] discussed implicit finite difference approximation for time fractional diffusion equation, Yang [9,10,11,12] studied local fractional derivative and obtained solution of fractional heat conduction problem by local fractional variation iteration method.

Shih et al. [8], Liu and Xu [2] gave analytic solution of pennes bio heat equation with sinusoidal heat flux condition on skin surface. Recently Ahmadikia et al. [1] gave the analytical solution of the parabolic and hyperbolic heat transfer equation with constant and transient heat flux condition on skin tissue. It is difficult to obtain an analytical solution of fractional bioheat equation.

In this paper, we give numerical solution of fractional bioheat equation with constant and sinusoidal heat flux condition on skin surface. We consider time fractional derivative of order $\alpha \in (0,1]$, which is in the form of Caputo fractional derivative and applying quadrature formula [5] on it. We use implicit finite difference method to solve the fractional bioheat model. The temperature profiles are obtained for different values of α, to study the effect of α on temperature profile in skin tissue.

2. Heat Transfer Model

2.1. Governing Equation

Pennes bioheat model [6] is implemented to study the heat transfer in skin tissue.

$$\rho c \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} + W_b c_b \left(T_a - T\right) + q_{met} \tag{1}$$

where $\rho c, k, T, t, x, T_a, W_b \left(= \rho_b w_b\right)$ and q_{met} represents density, specific heat, thermal conductivity, temperature, time, distance , artillery temperature ,blood perfusion rate and metabolic heat generation in skin tissue respectively.

In this study, we consider fractional form of Pennes bioheat heat model by replacing time derivative by fractional order derivative. We introduce governing equation for fractional Pennes bioheat model as follows

$$\rho c \frac{\partial^\alpha T}{\partial t^\alpha} = K \frac{\partial^2 T}{\partial x^2} + W_b c_b \left(T_a - T\right) + q_{met} \tag{2}$$

where fractional time derivative is of Caputo form defined [7] as

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial t}(t-s)^{-\alpha}\,ds, \ for \quad 0<\alpha<1$$
$$= \frac{\partial u(x,t)}{\partial t}, \quad for \ \ \alpha = 1 \tag{3}$$

On setting $\alpha = 1$, equation (2) reduces to Pennes bioheat equation [6].

2.2. Initial and Boundary Condition

$$T(x,0) = T_a \tag{4}$$

$$-k\frac{\partial T}{\partial x}_{|x=L} = 0, \tag{5}$$

at $x = 0$ we consider two types of boundary conditions,

2.2.1. Constant Heat Flux Condition

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q_0, \tag{6}$$

where q_0 is the heat flux on the skin surface

On making dimensionless variables

$$\zeta = \sqrt{\frac{W_b c_b}{k}}x,\ \eta = \left(\frac{W_b c_b}{\rho_t c_t}\right)^{(1/\alpha)} t,$$
$$\theta = \left(\frac{T-T_a}{q_0}\right)\sqrt{kW_b c_b},\ \Psi = \frac{q_{met}}{q_0\sqrt{\frac{W_b c_b}{k}}}$$

the equations (2),(4),(5) and (6) reduce to equations (7) to (10),

$$\frac{\partial^{\alpha}\theta}{\partial\eta^{\alpha}} = \frac{\partial^2\theta}{\partial\zeta^2} - \theta + \Psi, \tag{7}$$

$$\theta(\zeta,0) = 0 \tag{8}$$

$$\left.\frac{\partial\theta}{\partial\zeta}\right|_{\zeta=0} = -1, \tag{9}$$

$$\left.\frac{\partial\theta}{\partial\zeta}\right|_{\zeta=\sqrt{\frac{W_b c_b}{k}}L} = 0 \tag{10}$$

2.2.2. Sinusoidal Heat Flux Condition

In this section, on taking the cosine heat flux condition on skin surface, equation (6), is replaced by,

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q_0 cos(\omega t), \tag{11}$$

where ω is the heating frequency,

On making dimensionless variables

$$\zeta = \sqrt{\frac{\omega\rho c}{k}}x,\ \eta = (\omega)^{(1/\alpha)}t,\ \theta = \left(\frac{T-T_a}{q_0}\right)\sqrt{k\omega\rho c}$$
$$C_1 = \frac{W_b c_b}{\omega\rho c},\ \Phi = \frac{q_{met}}{q_0\sqrt{\frac{\omega\rho c}{k}}}$$

equations (2), (4), (5) and (11) become (12) to (15) in the following form,

$$\frac{\partial^{\alpha}\theta}{\partial\eta^{\alpha}} = \frac{\partial^2\theta}{\partial\zeta^2} - C_1\theta + \Phi, \tag{12}$$

$$\theta(\zeta,0) = 0 \tag{13}$$

$$\left.\frac{\partial\theta}{\partial\zeta}\right|_{\zeta=\sqrt{\frac{\omega\rho c}{k}}L} = 0 \tag{14}$$

$$\left.\frac{\partial\theta}{\partial\zeta}\right|_{\zeta=0} = -cos(\eta) \tag{15}$$

3. Numerical Solution

To get the numerical solution using finite difference method, space and time domain are divided into m and n equal partition respectively.

$m = L/h,\ h = \Delta x$ and $x_i = i*h,\ i = 0,1,2,.......,m,$ $n = t/k,\ k = \Delta t\ \ t_j = j*k,\ j = 0,1,2,......n,$ and L is the length of the tissue, $\Delta\zeta$ and $\Delta\eta$ are dimensionless space and time step size respectively.

For numerical solution of equation (2) the Caputo fractional derivative is approximated at point (i,n) by quadrature formula [5] as follows,

$$\left(\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}}\right)_i^n = \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\frac{1}{k^{\alpha}}....$$
$$\sum_{j=1}^{n}(\theta_i^j - \theta_i^{j-1})[(n-j+1)^{(1-\alpha)} - (n-j)^{(1-\alpha)}] \tag{16}$$

Considering

$S_j^{(\alpha)} = j^{(1-\alpha)} - (j-1)^{(1-\alpha)}$ and $\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{(1-\alpha)}\frac{1}{(k^{\alpha})},$

the Caputo fractional derivative given by equation(16) becomes,

$$\left(\frac{\partial^{\alpha}\theta}{\partial t^{\alpha}}\right)_i^n = \sigma_{\alpha,k}\sum_{j=1}^{n}S_j^{(\alpha)}\left(\theta_i^{n-j+1} - \theta_i^{n-j}\right) \tag{17}$$

Now using above approximation and central difference formulae for space derivative in equation (7) we get,

$$\sigma_{\alpha,k}\sum_{j=0}^{n}S_j^{(\alpha)}\left(\theta_i^{n-j+1} - \theta_i^{n-j}\right)$$
$$= \frac{1}{(\Delta\zeta)^2}\left(\theta_{i-1}^n - 2\theta_i^n + \theta_{i+1}^n\right) - \theta_i^n + \Psi \tag{18}$$

On taking $\gamma = \frac{1}{(\Delta\zeta)^2}$ and further simplification, gives,

$$-\gamma\theta_{i-1}^n + (\sigma_{\alpha,k} + 2\gamma + 1)\theta_i^n - \gamma\theta_{i+1}^n$$
$$= \sigma_{\alpha,k}\theta_i^{n-1} - \sigma_{\alpha,k}\sum_{j=2}^{n}S_j^{(\alpha)}\left(\theta_i^{n-j+1} - \theta_i^{n-j}\right) + \Psi \tag{19}$$

Similarly for equation (12), we get following finite difference equation,

$$-\gamma\theta_{i-1}^{n} + (\sigma_{\alpha,k} + 2\gamma + C_1)\theta_i^n - \gamma\theta_{i+1}^n = \sigma_{\alpha,k}\theta_i^{n-1} - \sigma_{\alpha,k}\sum_{j=2}^{n} S_j^{(\alpha)}\left(\theta_i^{n-j+1} - \theta_i^{n-j}\right) + \Phi \quad (20)$$

Initial condition and boundary condition at $x = L$ can be written as

$$\theta_i^0 = 0,\ i = 0,1,2,\ldots m, \quad (21)$$

$$\theta_m^j - \theta_{m-1}^j = 0,\ \ j = 0,1,2,\ldots n \quad (22)$$

Finite difference approximation of boundary condition at $x = 0$ for constant heat flux and sinusoidal heat flux given by equation (9) and (14) respectively can written as equation (23) and (24) respectively.

$$\theta_1^n - \theta_0^n = -\Delta\zeta; \quad (23)$$

$$\theta_1^n - \theta_0^n = -\Delta\zeta\, cos(\eta_j); \quad (24)$$

4. Results and Discussion

In this study we consider the following parameter values $L = 0.02\,m,\ \omega = 0.05\,sec^{-1}, T_a = 37°C,$ $q_0 = 5000\,W/m^2$, $\rho = 1050 kgm^{-3}$, $\rho_b = 1000 kgm^{-3}$, $q_{met} = 368.1\,Wm^{-3}$, $W_b = 0.5\,kgm^{-3}$, $c_b = 3770\ Jkg^{-1}C$, $c = 4180\ \ Jkg^{-1}C\ and\ K = 0.5\ Wm^{-1}C$.

We investigate the effect of anomalous diffusion in this model for different values of α (order of the time derivative).

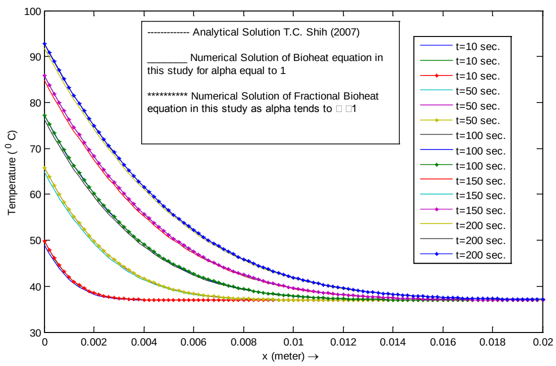

Figure 1. Comparison of analytic solution with numerical solution

4.1. Code Verification

Figure 1 represents the temperature profile along the distance for different time. This represents the comparison of analytic solution obtained by Shih et al. [8] and numerical solution for Pennes equation as well as numerical solution for fractional bioheat equation obtained by us as taking $\alpha \to 1$. We observe that our result is very similar to the analytic solution [8] which verifies with the developed computer code by us.

4.2. Effect of Fractional Order Derivative

Figure 2 shows that the temperature profile for constant heat flux along time at $x = 0m$ and $x = 0.001$m for different α . This is interesting to note that there is an elevation in temperature with decreased value of α , further decrease in temperature is observed with increase in depth of skin tissue. We also obtain the effect of sinusoidal heat flux on the skin surface as shown in Figure 3, Figure 4 and Figure 5. Figure 3 represents the temperature response at skin surface for different values of α. We also observe that amplitude increases with increase in α.

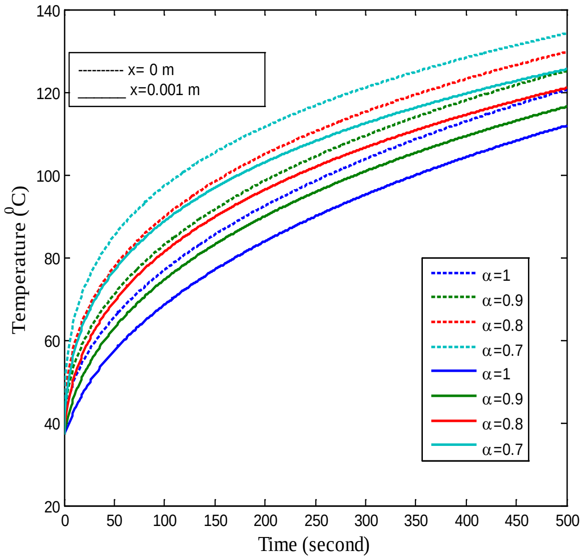

Figure 2. Temperature variation with respect to time for constant heat flux

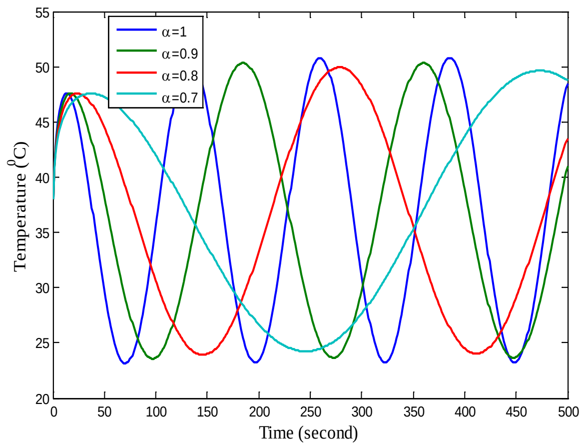

Figure 3. Temperature variation with respect to time for transient heat flux

Figure 4 shows that how much the sinusoidal heat flux effects the amplitude of dimensionless temperature (θ) along dimensionless time (η) at $x = 0$ m and $x = 0.001$ m, it shows that the amplitude for $x = 0.001$ m is higher than at $x = 0$ m. It is also observed that oscillation is decreasing as α decreases. Figure 5 indicates the dimensionless temperature profile (θ) along with dimensionless distance (ζ) for time $t = 60$ s and $t = 120$ s. The dimensionless temperature profile is substantially effected by the sinusoidal heat flux near $\zeta = 0$. However the oscillation of θ is very low beyond $\zeta = 5$. We see that the oscillation becomes low when the value of α decreases.

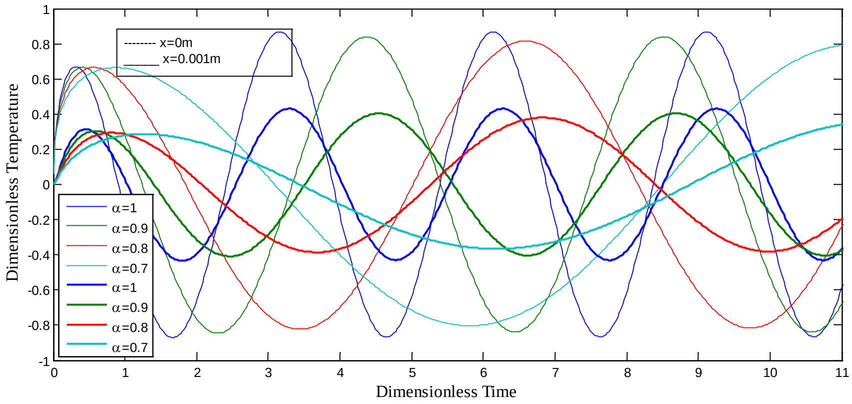

Figure 4. Dimensionless temperature profile with respect to dimensionless time transient heat flux

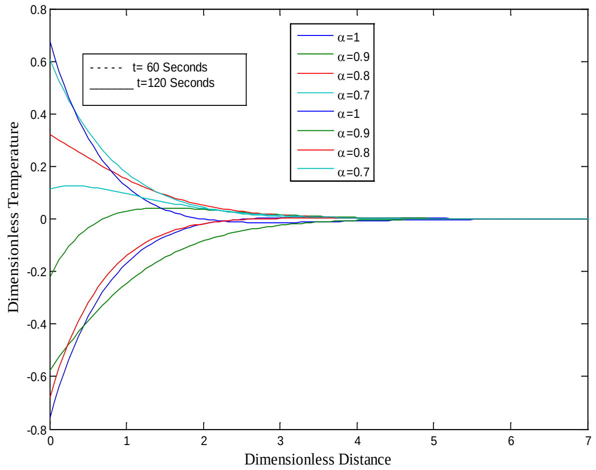

Figure 5. Dimensionless temperature profile with respect to dimensionless distance transient heat flux

5. Conclusion

In present paper the temperature distribution of fractional bioheat model has been studied with constant and sinusoidal heat flux on skin surface. The temperature profile has been obtained along with distance and time by considering constant heat flux and sinusoidal heat flux conditions. The effect of the anomalous diffusion has also been compared with normal diffusion for heat transfer in skin tissue. It has been observed that the temperature in fractional bioheat model with constant heat flux the becomes higher as compared to Pennes bioheat model. Further it increases when α decreases. It is also observed that if distance of the tissue increases then temperature decreases.

In fractional bioheat model with sinusoidal heat flux, oscillation decreases when α increases. It is noted that if tissue depth increases then amplitude decreases. We further observed the effect of α on temperature distribution which indicates that when α decreases the oscillation also reduces along the depth of the skin tissue.

The obtained results in this study may be useful to predict the temperature response in fractional bioheat model with constant and sinusoidal heat flux. The obtained solution may also be useful for experimental model to predict the value of α . This may be applicable for thermal therapy in medical sciences.

Nomenclature

ρ	density (kg/m^3)
c	specific heat $(J/kg°C)$
k	thermal conductivity of the tissue $(W/m°C)$
W_b	blood perfusion rate $(Kg/m^3 s)$
q_0	volumetric heat generation (W/m^3)
ω	heating frequency$(\sec^{-1})$
T	temperature (°C)
t	time (sec)
x	space coordinate(m)
θ	dimensionless Temperature
ζ	dimensionless Space
η	dimensionless Time
b	subscript for blood

References

[1] Ahmadikia H., Fazlali R. and Moradi A., "Analytic solution of the parabolic and hyperbolic heat transfer equations with constant and transient heat flux conditions on skin tissue", *International communications in heat and mass transfer*, 39 (2012), 121-130.

[2] Liu J., Xu L.X., "Estimation of blood perfusion using phase shift in temperature response to sinusoidal heating the skin surface", *IEEE Transaction on biomedical engineering*, 46 (1999), 1037-43.

[3] Meerschaert M. M. and Tadjeran C., "Finite difference approximations for fractional advection-dispersion flow equations", *Journal of Computational and Applied Mathematics*, 172(2004), 65-77.

[4] Meerschaert M. M., Scheffler H.P. and Tadjeran C., "Finite difference methods for two-dimensional fractional dispersion equations", *Journal of Computational Physics*, 211(2006), 249-261.

[5] Murio D. A., "Implicit finite difference approximation for time fractional diffusion equations", *Computers and mathematics with Applications*, 56 (2008), 1138-1145.

[6] Pennes H. H., "Analysis of tissue and arterial blood temperature in the resting forearm", *Journal of applied physiology*, 1(1948), 93-122.

[7] Podlubny I., *Fractional Differential Equations*, Academic Press, New York (1999).

[8] Shih T. C. , Yuan P., Lin W. L. and Kou H. S. ," Analytical analysis of the Pennes bioheat transfer equation with sinusoidal heat flux condition on skin surface", *Medical engineering physics*, 29(2007), 946-953.

[9] Yang X.J. *advanced local fractional calculus and its applications*, World science publisher, New York, USA, 2012.

[10] Yang X.J. and Baleanu D., “Fractional heat conduction problem solved by local fractional variation iteration method”, *Thermal science*, 17(2), 2013, 625-628.

[11] Yang X.J. *Local Fractional Functional Analysis and Its Applications*, Asia Academic Publisher Ltd. Hong Kong China, 2011.

[12] Hu M. S., Baleanu D and Yang X.J. “One-Phase Problems for Discontinuous Heat Transfer in Fractal Media”, *Mathematical Problems in Engineering*, Volume 2013 Article ID 358473, (2013).