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A Mixed Force-Displacement Method for the Exact Solution of Plane Frames

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Abstract This paper deals with the solution of statically undetermined plane frames by using a mixed force-displacement method based on the use of the differential equations of both the bar axial deformation and the beam bending. The unknowns in solving the algebraic equations derived by the proposed approach are represented by the integration constants of each mono-axial frame of the structure. The applications examples show that, even if the dimensions of the problem are larger than both cases related to the use of the force and of the displacement methods, the proposed approach does not require post-processing for finding any kinematic and static response quantity. Therefore, this approach can be considered as an alternative to the Finite Element approaches for solving plane multi-axial frames.

Keywords: *statically indeterminate structures, plane frames, beam bending differential equation, bar axial deformation*

SIGN CONVENTION

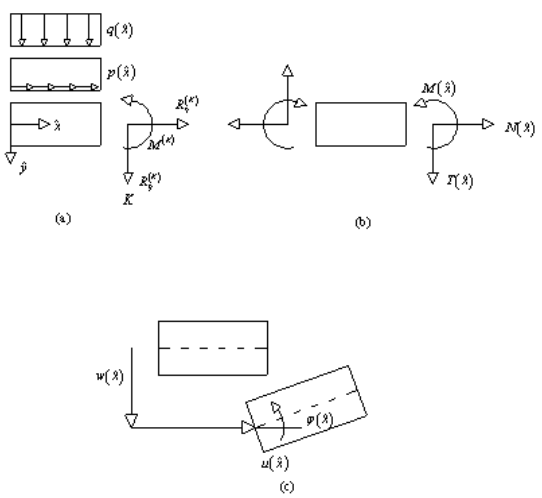

Figure 1. Sign convention for: a) applied distributed loads and reactions; b) internal forces; c) displacements.

1. Introduction

The methods for finding the solution, in terms of both kinematic and static quantities, of statically indeterminate plane frames are fundamental topics of Structural Mechanics. In the literature, the most treated approaches are the force and the displacement methods [1,2,3,4]. The use of the first one is preferable when the frame shows a low number of static indeterminations, while the second one is better if the number of deformation modes of the frame, in a discretized system, is low.

When the frame is mono-axial a suitable approach for its solution is considered by the use of the bar-axial deformation and the beam-bending differential equations [1,2]. In general, it requires the determination of 4+2=6 integration constants that must be obtained by imposing a corresponding number of boundary conditions.

The pioneering works by Macaulay [5], Brungraber [6], and the work by Falsone [7] generalized the use of this approach to all the cases in which some singularities are present in these equations. They are due to some particular loading conditions or to the presence of some natural and/or essential constraints along the axis of the frame. In this way the application of the differential equation approach can be considered as a general tool for solving the mono-axial frames. No extension of this approach to multi-axial frames appears to be in the literature. As a consequence, the only true alternative to the force and displacement approaches for the solution of multi-axial frames is the Finite Element approach.

The aim of this paper is to extend the differential equation approach to the case of multi-axial frames. In particular, it will be shown that this "global" method is suitable for a matrix notation. As a matter of fact, each matrix quantity can be easily expressed in terms of the geometry of the structure, the presence of the various constraints and the material properties. The number of unknowns is always equal to $6\times n$, n being the number of mono-axial frames composing the structure. This number is higher than those related to the use of the force method, the displacements method and the Finite Element approach (if the discretization is not too dense). However, the proposed method has the advantage that the knowledge of the $6\times n$ integration constants allows immediately defining any kinematic and static response, without the necessity of a post-processing work, necessary, for example, when the Finite Element approach is applied.

2. Bar Axial and Beam Bending Differential Equations

The method presented in this work for the exact solution of linearly elastic statically indeterminate frames is essentially based on the use of the classical differential equations governing the axial behavior of the bar and the deflection behavior of the beam. For this reason these two equations are treated in this section, with further objective to introduce the fundamental symbolism used in this work.

The differential equation governing the axial generalized displacement, $u(x)$, of a homogeneous elastic bar, with constant axial stiffness, EA, where E is the Young modulus and A is the cross-section area, has the following form:

$$u''(x) = -\frac{1}{EA} p(x) \tag{1}$$

where the apex indicates the derivative with respect to x and $p(x)$ is the axial continuous load acting on the bar. It is important to note that, if some concentrated axial forces act on the beam, Eq. (1) remains valid if the generalized functions are used as in [7].

A first integration of Eq. (1) leads to:

$$u'(x) = -\frac{1}{EA} p^{(1)}(x) + C_1 \;\Rightarrow\; \varepsilon(x) = u'(x) = -\frac{1}{EA} p^{(1)}(x) + C_1 \tag{2}$$

where the apex inside the brackets indicates the order of integration made on the function, $\varepsilon(x)$ is the generalized axial strain and C_1 is an integration constant. In the second of these equations the compatibility equation between axial displacement and deformation has been taken into consideration. By considering the constitutive relationship, the axial internal force $N(x)$ can be easily found as:

$$N(x) = EA\varepsilon(x) = -p^{(1)}(x) + EAC_1 \tag{3}$$

A further integration of Eq. (2) gives:

$$u(x) = -\frac{1}{EA} p^{(2)}(x) + C_1 x + C_2 \tag{4}$$

The values of the constants C_1 and C_2 are obtainable by imposing the boundary conditions that can be of static type (based on the value of the internal axial force) and/or kinematic type (based on the value of the axial displacement).

It is important to note that in both cases of statically determinate and indeterminate bars, imposition of the boundary conditions leads to a system of two equations with two unknowns admitting a unique solution. On the contrary, when the bar is kinematically indeterminate (which happens when the boundary conditions are both of static type), the algebric solution is impossible.

The differential equation governing the deflection generalized displacement, $w(x)$, of a homogeneous elastic Bernoulli beam, with constant axial stiffness, EI, I being the cross-section inertia moment, has the following form:

$$w''''(x) = \frac{1}{EI} q(x) \tag{5}$$

where $q(x)$ is the transversal continuous load acting on the beam. The use of the generalized function allows considerations of the cases of discontinuous loads, and of presence of internal and external constrains along the axis [7].

A first integration of Eq. (5) leads to:

$$w'''(x) = \frac{1}{EI} q^{(1)}(x) + C_3 \tag{6}$$

The following relationship holds in the case of the Euler-Bernoulli beam:

$$T(x) = -EIw'''(x) \tag{7}$$

$T(x)$ being the internal shear force. Substituting from Eq.(6) gives:

$$T(x) = -q^{(1)}(x) - C_3 EI \tag{8}$$

While the integration of Eq. (6) gives:

$$w''(x) = \frac{1}{EI} q^{(2)}(x) + C_3 x + C_4 \;\Rightarrow\; M(x) = -q^{(2)}(x) - EI\left(C_3 x + C_4\right) \tag{9}$$

where the relationship between the beam bending moment $M(x)$ and the deflection of the Euler-Bernoulli beam, that is $M(x) = -EIw''(x)$, has been considered.

Further integration of Eq. (9) leads to:

$$w'(x) = \frac{1}{EI} q^{(3)}(x) + C_3 \frac{x^2}{2} + C_4 x + C_5 \;\Rightarrow\; \varphi(x) = -\left(\frac{1}{EI} q^{(3)}(x) + C_3 \frac{x^2}{2} + C_4 x + C_5\right) \tag{10}$$

where the compatibility relationship of the Euler-Bernoulli beam between the generalized rotation, $\varphi(x)$ and the deflection, that is $\varphi(x) = -w'(x)$, has been taken into consideration.

Finally, the last integration of Eq. (10) gives:

$$w(x) = \frac{1}{EI} q^{(4)}(x) + C_3 \frac{x^3}{6} + C_4 \frac{x^2}{2} + C_5 x + C_6 \tag{11}$$

The four integration constants $C_3 \cdots C_6$ can be evaluated by imposing the four boundary conditions of the beam. In the cases of statically determinate and indeterminate beams the algebraic four equations resulting by this imposition define a system admitting only one solution. Instead, in the case of unstable frames the system of algebraic solution is impossible.

When a mono-axial frame is considered, then its solution can be obtained by considering together, but even separately being independent, the differential equation governing the axial deformation and the differential equation governing the beam bending. This means that six integration constants must be evaluated by imposing the six boundary conditions. The corresponding system of six algebraic equations gives again a unique solution for statically determinate and indeterminate frames, while the solution is impossible when the frame is unstable.

3. Mixed Force-Displacement Method for Frames

A generic frame can be considered as a system of mono-axial frames constrained externally in correspondence of the *external nodes* and internally, each other, in correspondence of the *internal nodes*. Figure.2a shows one of the simplest frames, while in Figure 2b a more generic frame is represented. Here, for simplicity, it is assumed that all the constraints, both external and internal, are fixed joints, even if the method can be easily rearranged for other types of constraints.

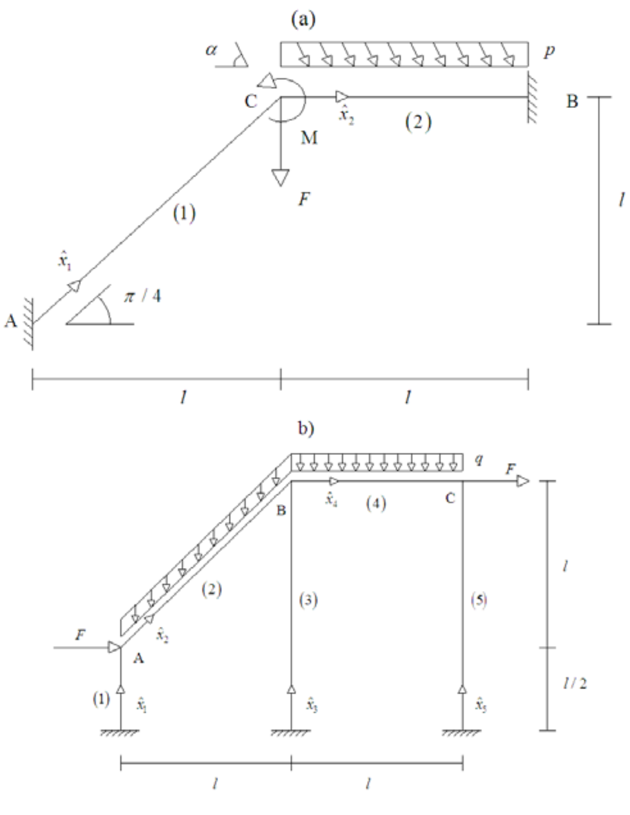

Figure 2. a) simple frame; b) generic frame

The generic i-th mono-axial element of a frame, referred to a local axis system in which the axis $\hat{x}_i$ coincides with its axis, is loaded by a generic axial distributed load, $p(\hat{x}_i)$, and by a generic transversal distributed load $q(\hat{x}_i)$

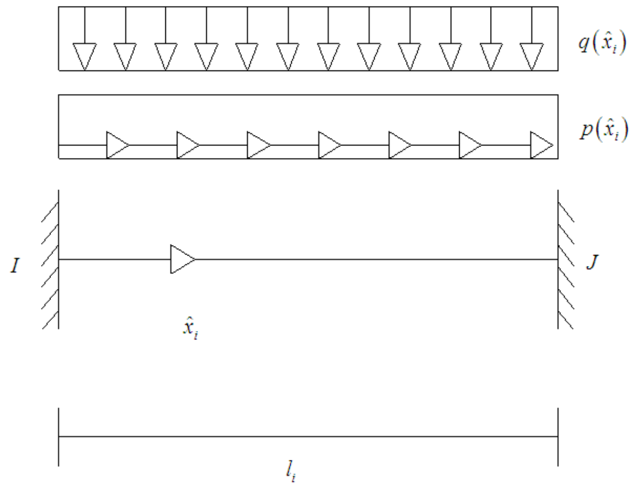

Figure 3. mono-axial frame

The behavior of this element is governed by the following two differential equations:

$$u''(\hat{x}_i) = -\frac{1}{(EA)_i} p(\hat{x}_i); \qquad w''''(\hat{x}_i) = \frac{1}{(EI)_i} q(\hat{x}_i) \tag{12, a-b}$$

whose successive integrations lead to the following solutions in terms of generalized displacements:

$$u(\hat{x}_i) = -\frac{1}{(EA)_i} p^{(2)}(\hat{x}_i) + C_{i,1}\hat{x}_i + C_{i,2}; \qquad w(\hat{x}_i) = \frac{1}{EI} q^{(4)}(\hat{x}_i) + C_{i,3}\frac{\hat{x}_i^3}{6} + C_{i,4}\frac{\hat{x}_i^2}{2} + C_{i,5}\hat{x}_i + C_{i,6} \tag{13,a-b}$$

In order to express the boundary conditions, it is important to find the generalized displacements, u, w and φ, and the constrain reactions, $R_{\hat{x}_i}$, $R_{\hat{y}_i}$ and M, at the extreme nodes, I and J, of the mono-axial frame, these are:

$$u_i^{(I)} = u(\hat{x}_i)|_{\hat{x}_i=0} = -\frac{1}{(EA)_i} p^{(2)}(\hat{x}_i)|_{\hat{x}_i=0} + C_{i,2};$$
$$w_i^{(I)} = w(\hat{x}_i)|_{\hat{x}_i=0} = \frac{1}{(EI)_i} q^{(4)}(\hat{x}_i)|_{\hat{x}_i=0} + C_{i,6}; \tag{14, a-c}$$
$$\varphi_i^{(I)} = -w'(\hat{x}_i)|_{\hat{x}_i=0} = -\frac{1}{(EI)_i} q^{(3)}(\hat{x}_i)|_{\hat{x}_i=0} - C_{i,5}$$

$$R_{\hat{x}_i}^{(I)} = -(EA)_i\, u'(\hat{x}_i)|_{\hat{x}_i=0} = p(\hat{x}_i)|_{\hat{x}_i=0} - (EA)_i\, C_{i,1};$$
$$R_{\hat{y}_i}^{(I)} = (EI)_i\, w'''(\hat{x}_i)|_{\hat{x}_i=0} = q^{(1)}(\hat{x}_i)|_{\hat{x}_i=0} + (EI)_i\, C_{i,3}; \tag{15, a-c}$$
$$M_i^{(I)} = (EI)_i\, w''(\hat{x}_i)|_{\hat{x}_i=0} = q^{(2)}(\hat{x}_i)|_{\hat{x}_i=0} + (EI)_i\, C_{i,4}$$

$$\begin{aligned} u_i^{(J)} &= u(\hat{x}_i)\,|_{\hat{x}_i=l_i} \\ &= -\frac{1}{(EA)_i}\, p^{(2)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} + l_i C_{i,1} + C_{i,2}; \\ w_i^{(J)} &= w(\hat{x}_i)\,|_{\hat{x}_i=l_i} \\ &= \frac{1}{(EI)_i}\, q^{(4)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} + \frac{l_i^3}{6} C_{i,3} + \frac{l_i^2}{2} C_{i,4} + l_i C_{i,5} + C_{i,6}; \\ \varphi_i^{(J)} &= -w'(\hat{x}_i)\,|_{\hat{x}_i=l_i} \\ &= -\frac{1}{(EI)_i}\, q^{(3)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} - \frac{l_i^2}{2} C_{i,3} - l_i C_{i,4} - C_{i,5} \end{aligned} \tag{16, a-c}$$

$$\begin{aligned} R_{\hat{x}_i}^{(J)} &= (EA)_i\, u'(\hat{x}_i)\,|_{\hat{x}_i=l_i} \\ &= -p(\hat{x}_i)\,|_{\hat{x}_i=l_i} + (EA)_i\, C_{i,1}; \\ R_{\hat{y}_i}^{(J)} &= -(EI)_i\, w'''(\hat{x}_i)\,|_{\hat{x}_i=l_i} \\ &= -q^{(1)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} - (EI)_i\, C_{i,3}; \\ M_i^{(J)} &= -(EI)_i\, w''(\hat{x}_i)\,|_{\hat{x}_i=l_i} \\ &= -q^{(2)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} - (EI)_i\, l_i C_{i,3} - (EI)_i\, C_{i,4} \end{aligned} \tag{17, a-c}$$

Each of the above mentioned four groups of relationships can be rewritten in matrix form as follows:

$$\begin{aligned} \hat{\mathbf{u}}_i^{(K)} &= \hat{\mathbf{A}}_i^{(K)} \mathbf{c}_i - \hat{\mathbf{a}}_i^{(K)}; \\ \hat{\mathbf{r}}_i^{(K)} &= \hat{\mathbf{B}}_i^{(K)} \mathbf{c}_i - \hat{\mathbf{b}}_i^{(K)} \qquad K = I, J \end{aligned} \tag{18, a-b}$$

where:

$$\begin{aligned} \hat{\mathbf{u}}_i^{(K)} &= \begin{pmatrix} u_i^{(K)} & w_i^{(K)} & \varphi_i^{(K)} \end{pmatrix}^T; \\ \hat{\mathbf{r}}_i^{(K)} &= \begin{pmatrix} R_{\hat{x}_i}^{(K)} & R_{\hat{y}_i}^{(K)} & M_i^{(K)} \end{pmatrix}^T \qquad K = I, J \end{aligned} \tag{19, a-b}$$

$\mathbf{c}_i$ collects the six integration constants of the i-th mono-axial frame and $\hat{\mathbf{A}}_i^{(K)}$ and $\hat{\mathbf{B}}_i^{(K)}$ are expressed as follows:

$$\hat{\mathbf{A}}_i^{(I)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}; \qquad \hat{\mathbf{A}}_i^{(J)} = \begin{pmatrix} l_i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{l_i^3}{6} & \frac{l_i^2}{2} & l_i & 1 \\ 0 & 0 & -\frac{l_i^2}{2} & -l_i & -1 & 0 \end{pmatrix} \tag{20, a-b}$$

$$\hat{\mathbf{B}}_i^{(I)} = \begin{pmatrix} -(EA)_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (EI)_i & 0 & 0 & 0 \\ 0 & 0 & 0 & (EI)_i & 0 & 0 \end{pmatrix}; \qquad \hat{\mathbf{B}}_i^{(J)} = \begin{pmatrix} (EA)_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(EI)_i & 0 & 0 & 0 \\ 0 & 0 & -(EI)_i\, l_i & -(EI)_i & 0 & 0 \end{pmatrix} \tag{21, a-b}$$

Finally the vectors $\hat{\mathbf{a}}_i^{(K)}$ and $\hat{\mathbf{b}}_i^{(K)}$, which depend on the external loads, can be expressed as follows:

$$\begin{aligned} \hat{\mathbf{a}}_i^{(I)} &= \begin{pmatrix} \frac{1}{(EA)_i} p^{(2)}(\hat{x}_i)\,|_{\hat{x}_i=0} & -\frac{1}{(EI)_i} q^{(4)}(\hat{x}_i)\,|_{\hat{x}_i=0} & \frac{1}{(EI)_i} q^{(3)}(\hat{x}_i)\,|_{\hat{x}_i=0} \end{pmatrix}^T; \\ \hat{\mathbf{a}}_i^{(J)} &= \begin{pmatrix} \frac{1}{(EA)_i} p^{(2)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} & -\frac{1}{(EI)_i} q^{(4)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} & \frac{1}{(EI)_i} q^{(3)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} \end{pmatrix}^T \end{aligned} \tag{22, a-b}$$

$$\begin{aligned} \hat{\mathbf{b}}_i^{(I)} &= \begin{pmatrix} -p^{(1)}(\hat{x}_i)\,|_{\hat{x}_i=0} & -q^{(1)}(\hat{x}_i)\,|_{\hat{x}_i=0} & -q^{(2)}(\hat{x}_i)\,|_{\hat{x}_i=0} \end{pmatrix}^T; \\ \hat{\mathbf{b}}_i^{(J)} &= \begin{pmatrix} p^{(1)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} & q^{(1)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} & q^{(2)}(\hat{x}_i)\,|_{\hat{x}_i=l_i} \end{pmatrix}^T \end{aligned} \tag{23, a-b}$$

When the external nodes are fixed, the compatibility conditions imply that all the displacements are zero in these points. For example, for the frame represented in Figure 2a it should be:

$$\begin{aligned} \hat{\mathbf{u}}^{(A)} \equiv \hat{\mathbf{u}}_1^{(I)} = \mathbf{0} \quad &\Rightarrow \quad \hat{\mathbf{A}}_1^{(I)} \mathbf{c}_1 = \hat{\mathbf{a}}_1^{(I)} \\ \hat{\mathbf{u}}^{(B)} \equiv \hat{\mathbf{u}}_2^{(J)} = \mathbf{0} \quad &\Rightarrow \quad \hat{\mathbf{A}}_2^{(J)} \mathbf{c}_2 = \hat{\mathbf{a}}_2^{(J)} \end{aligned} \tag{24, a-b}$$

These relationships represent a system of six scalar algebraic equations in the twelve unknowns defined by the components of the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$. The remaining six equations, necessary for solving the system, must be related to the compatibility and equilibrium conditions of the free node *C*.

In order to write these equations, firstly it is necessary to refer all the static and kinematic employed quantities to the same global reference axis system. In this operation the introduction of the so-called *rotation matrix*, $\mathbf{G}_i$, is necessary for passing from the definition of any vector, $\hat{\mathbf{v}}$, referred to the local axis system $(O; \hat{x}_i, \hat{y}_i)$, to the same vector, $\mathbf{v}$, referred to the global axis system $(O; x, y)$. If α_i is the angle between $\hat{x}_i$ and x, this matrix has the following form:

$$\mathbf{G}_i = \begin{pmatrix} \cos\alpha_i & \sin\alpha_i & 0 \\ -\sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

therefore:

$$\mathbf{v} = \mathbf{G}_i \hat{\mathbf{v}} \tag{26}$$

The compatibility conditions of the free node *C* of the frame of Figure 2a gives:

$$\begin{aligned} &\mathbf{u}_1^{(J)} = \mathbf{u}_2^{(I)} \quad \Rightarrow \quad \mathbf{G}_1 \hat{\mathbf{u}}_1^{(J)} = \mathbf{G}_2 \hat{\mathbf{u}}_2^{(I)} \\ &\Rightarrow \quad \mathbf{G}_1 \hat{\mathbf{A}}_1^{(J)} \mathbf{c}_1 - \mathbf{G}_2 \hat{\mathbf{A}}_2^{(I)} \mathbf{c}_2 = \mathbf{G}_1 \hat{\mathbf{a}}_1^{(J)} - \mathbf{G}_2 \hat{\mathbf{a}}_2^{(I)} \\ &\Rightarrow \quad \mathbf{A}_1^{(J)} \mathbf{c}_1 - \mathbf{A}_2^{(I)} \mathbf{c}_2 = \mathbf{a}_1^{(J)} - \mathbf{a}_2^{(I)} \end{aligned} \tag{27}$$

The last three equations necessary for solving the frame problem under examination are related to the equilibrium conditions of the free node *C*:

$$\begin{aligned} &\mathbf{r}_1^{(J)} + \mathbf{r}_2^{(I)} = \mathbf{f}^{(C)} \quad \Rightarrow \quad \mathbf{G}_1 \hat{\mathbf{r}}_1^{(J)} + \mathbf{G}_2 \hat{\mathbf{r}}_2^{(I)} = \mathbf{f}^{(C)} \\ &\Rightarrow \quad \mathbf{G}_1 \hat{\mathbf{B}}_1^{(J)} \mathbf{c}_1 + \mathbf{G}_2 \hat{\mathbf{B}}_2^{(I)} \mathbf{c}_2 = \mathbf{G}_1 \hat{\mathbf{b}}_1^{(I)} + \mathbf{G}_2 \hat{\mathbf{b}}_2^{(J)} + \mathbf{f}^{(C)} \\ &\Rightarrow \quad \mathbf{B}_1^{(J)} \mathbf{c}_1 + \mathbf{B}_2^{(I)} \mathbf{c}_2 = \mathbf{b}_1^{(I)} + \mathbf{b}_2^{(J)} + \mathbf{f}^{(C)} \end{aligned} \tag{28}$$

in which $\mathbf{f}^{(C)}$ collects the concentrated loads applied directly on the node *C*.

Reassuming, the solving system of equations is composed by Eqs. (24), (27) and (28) and can be written in the following compact form:

$$\mathbf{Dc} = \mathbf{d} \tag{29}$$

where:

$$\mathbf{D} = \begin{pmatrix} \mathbf{A}_1^{(I)} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2^{(J)} \\ \mathbf{A}_1^{(J)} & -\mathbf{A}_2^{(I)} \\ \mathbf{B}_1^{(J)} & \mathbf{B}_2^{(I)} \end{pmatrix}; \quad \mathbf{d} = \begin{pmatrix} \mathbf{a}_1^{(I)} \\ \mathbf{a}_2^{(J)} \\ \mathbf{a}_1^{(J)} - \mathbf{a}_2^{(I)} \\ \mathbf{b}_1^{(J)} + \mathbf{b}_2^{(I)} + \mathbf{f}^{(C)} \end{pmatrix}; \quad \mathbf{c} = \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix} \tag{30, a-c}$$

It is important to note that in reporting the equations in the form of Eq. (29), both the members of Eqs. (24) have been pre-multiplied by the rotation matrices $\mathbf{G}_1$ and $\mathbf{G}_2$, respectively. Obviously, this last operation does not change the result.

Like the cases of the bars, the beams and the mono-axial frames, even for the case of multi-axial frames, have only one solution for statically determined and undetermined systems. Hence the square matrix $\mathbf{D}$ is invertible. The inverse of Eq. (29) gives the value of the constant vector $\mathbf{c}$ and from this value all the exact values for static and kinematic quantities defining the system response can be immediately obtained.

In a following section the system represented in Figure 2a will be solved as an example.

4. Automatic Construction of the Solving System

The relationship given in Eq. (29) can be extended to any generic multi-axial frame. However, the application of the mixed method could be improved if a direct construction of the quantities of the Eq. (29) can be achieved.

For this purpose, it is easy to show that the construction of the vector $\mathbf{c}$ is immediate: it is a vector of order $6 \cdot n$, if *n* is the number of mono-axial frames composing the structure, collecting the integration constant vectors $\mathbf{c}_i$ of each mono-axial frame ($i = 1, \cdots, n$).

The matrix $\mathbf{D}$, of order $6 \cdot n \times 6 \cdot n$, is built by matrices of order 3×6, while the vector $\mathbf{d}$, of order $6 \cdot n$, is built by $2n$ vectors of order 3. Some row-blocks of $\mathbf{D}$ refer to the external constrained nodes of the structure. In particular, if the extreme point *K* of the *i*-th element is fixed, then a row-block of $\mathbf{D}$ must show all zero matrices, except for the block corresponding to the *i*-th column-block, where the matrix $\mathbf{A}_i^{(K)}$ must be inserted. It is worth remembering that if the node considered is the first one (respect to the local axes), then $K \equiv I$; otherwise, it is $K \equiv J$. In the corresponding row-block of $\mathbf{d}$, the vector $\mathbf{a}_i^{(K)}$ must be inserted.

Some other row-blocks of $\mathbf{D}$ are related to the compatibility conditions of the internal free nodes: if *m* mono-axial frames are connected in a node, then $m-1$ row blocks correspond to these conditions; in each of these row-blocks only two matrices are not zero, these are those corresponding to two of the elements connected in the node, placed in correspondence of the column-blocks corresponding to the elements. If, for example, the mono-axial frame elements are the *j*-th and the *k*-th ones, the non-zero matrices are $\mathbf{A}_j^{(K)}$ and $-\mathbf{A}_k^{(K)}$, placed in the *j*-th and in the *k*-th block-column, respectively. In correspondence of the same row-block, in the vector $\mathbf{d}$ the vector $\mathbf{a}_j^{(K)} - \mathbf{a}_k^{(K)}$ must be placed. The couple of elements to be considered in each of the $m-1$ row-blocks related to the node in examination must be chosen in such a way that no repeated compatibilities arise (and, consequently, no loosing ones are).

At last, the remaining row-blocks correspond to the equilibrium conditions in the internal free nodes. Each row-block corresponds to a node: if *m* elements are connected in the node *C*, zero matrices are placed in correspondence of the column-blocks related to elements not connected in *C*, while the matrices $\mathbf{B}_j^{(K)}$ must be placed in the cross of all the *m* column-blocks corresponding to the elements connected in the node. In the same row-block, in the vector $\mathbf{d}$ the vector $\mathbf{f}^{(C)} + \sum_m \mathbf{b}_i^{(K)}$ .must be allocated

In the example section the vector and matrix quantities necessary for building the solution equation of the multi-axial frames represented in Fig.2b are reported.

The proposed method is clearly very suitable for the implementation in a program calculus.

5. Examples

In this section some examples are given in order to better clarify the proposed approach and its practical application.

5.1. Mono-axial Frame Samples

In this sub-section the two mono-axial frames represented in Figure 3 are considered. The expressions of the axial displacement and of the transversal deflection are given by the specifications of Eqs. (4) and (11):

$$\begin{aligned} u(x) &= -\frac{q \sin\alpha}{EA}\frac{x^2}{2} + C_1 x + C_2; \\ w(x) &= \frac{q \cos\alpha}{EI}\frac{x^4}{24} + C_3\frac{x^3}{6} + C_4\frac{x^2}{2} + C_5 x + C_6 \end{aligned} \tag{31, a-b}$$

The first frame of these figures is characterized by the following boundary conditions:

$$
\begin{aligned}
&u(x)|_{x=0}=0 \;\Rightarrow\; C_2=0; \quad w(x)|_{x=0}=0 \;\Rightarrow\; C_6=0;\\
&\varphi(x)|_{x=0}=0 \;\Rightarrow\; w'(x)|_{x=0}=0 \;\Rightarrow\; C_5=0;\\
&u(x)|_{x=l}=0 \;\Rightarrow\; -\frac{q\cos\alpha}{EA}\frac{l^2}{2}+C_1 l+C_2=0;\\
&w(x)|_{x=l}=0 \;\Rightarrow\; \frac{q\sin\alpha}{EI}\frac{l^4}{24}+C_3\frac{l^3}{6}+C_4\frac{l^2}{2}+C_5 l+C_6=0;\\
&\varphi(x)|_{x=l}=0 \;\Rightarrow\; w'(x)|_{x=l}=0 \;\Rightarrow\; \frac{q\sin\alpha}{EI}\frac{l^3}{6}+C_3\frac{l^2}{2}+C_4 l+C_5=0
\end{aligned}
\qquad (32,\text{ a-f})
$$

this define a system of six equations in the six unknowns $C_1,\cdots,C_6$ yielding the following unique solution:

$$
\begin{aligned}
&C_1=\frac{ql\cos\alpha}{2EA};\\
&C_2=0;\\
&C_3=-\frac{ql\sin\alpha}{2EI};\\
&C_4=\frac{ql^2\sin\alpha}{12EI};\\
&C_5=0;\\
&C_6=0
\end{aligned}
\qquad (33,\text{ a-f})
$$

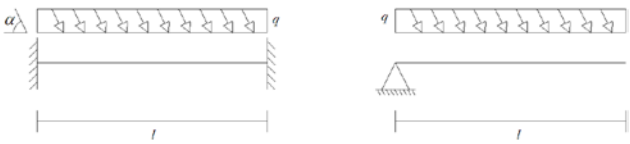

Figure 3. mono-axial frame examples: a) statically undetermined; b) unstable

At this point all the displacements and internal forces can be obtained immediately. It is important to note that the integration constants C_1 and C_2 appear only in the two Eqs. (40,a,d). This means that they can be obtained by solving simply a system of two equations and two unknowns. The other four constants can be obtained by solving the other four equations. Obviously this is because of the independence between the axial and the deflection problems.

The second frame considered in this subsection is the unstable one represented in Figure 3b. Eqs.(31) are still valid, while the corresponding boundary conditions are:

$$
\begin{aligned}
&u(x)|_{x=0}=0 \;\Rightarrow\; C_2=0; \quad w(x)|_{x=0}=0 \;\Rightarrow\; C_6=0;\\
&M(x)|_{x=0}=0 \;\Rightarrow\; w''(x)|_{x=0}=0 \;\Rightarrow\; C_4=0;\\
&N(x)|_{x=l}=0 \;\Rightarrow\; u'(x)|_{x=l}=0 \;\Rightarrow\; -\frac{q\cos\alpha}{EA}l+C_1=0;\\
&M(x)|_{x=l}=0 \;\Rightarrow\; w''(x)|_{x=l}=0 \;\Rightarrow\; \frac{q\sin\alpha}{EI}\frac{l^2}{2}+C_3 l+C_4=0;\\
&T(x)|_{x=l}=0 \;\Rightarrow\; w'''(x)|_{x=l}=0 \;\Rightarrow\; \frac{q\sin\alpha}{EI}l+C_3=0
\end{aligned}
\qquad (34,\text{a-f})
$$

this shows an impossible solving system. However, it is important to note that, if only the two Eqs. (32,a,d) are considered, they are able to give a solution for the constant C_1 and C_2. This is due to the fact that the axial problem related to the frame under examination is statically determined.

5.2. Multi-axial Frame Samples

Two samples are treated in this sub-section: the first one is the very simple example represented in Figure 2a, for which all the matrix quantities, necessary for the solution, are given in section 3. In particular, it is necessary to consider the solving equation having the form given into Eq. (29), where the matrix $\mathbf{D}$ and the vectors $\mathbf{c}$ and $\mathbf{d}$ are given in Eqs. (30).

As the node I of the element 1 and the node J of the element 2 are externally fixed, the matrices $\hat{\mathbf{A}}_1^{(I)}$ and $\hat{\mathbf{A}}_2^{(J)}$, referred to the local reference axes, can be used instead of the matrices $\mathbf{A}_1^{(I)}$ and $\mathbf{A}_2^{(J)}$, respectively. They have the detailed expressions of Appendix (1). In the same appendix the other vectors and matrices used in Eqs. (29) and (30) are given. Moreover, the values of the integration are obtained there.

The values of these constants given in appendix (1) imply the following forms of the kinematic and static laws of the two mono-axial frames:

$$
\begin{aligned}
&u(\hat{x}_1)=-0.1626\times10^{-4}\hat{x}_1;\\
&w(\hat{x}_1)=-2.0312\times10^{-4}\frac{\hat{x}_1^3}{6}+4.8669\times10^{-4}\frac{\hat{x}_1^2}{2}\\
&\varphi(\hat{x}_1)=2.0312\times10^{-4}\frac{\hat{x}_1^2}{2}-4.8669\times10^{-4}\hat{x}_1;\\
&N(\hat{x}_1)=1.6271\times10^{-5}EA\\
&M(\hat{x}_1)=-\left(-2.0312\times10^{-4}\hat{x}_1+4.8669\times10^{-4}\right)EI;\\
&T(\hat{x}_1)=2.0312\times10^{-4}EI
\end{aligned}
$$

$$
\begin{aligned}
&u(\hat{x}_2)=-\frac{p\cos\alpha}{EA}\frac{\hat{x}_2^2}{2}-0.09583\times10^{-4}\hat{x}_2+0.5896\times10^{-4};\\
&w(\hat{x}_2)=\frac{p\sin\alpha}{EI}\frac{\hat{x}_2^4}{24}-5.3723\times10^{-4}\frac{\hat{x}_2^3}{6}+13.9417\times10^{-4}\frac{\hat{x}_2^2}{2}\\
&\qquad -16.3655\times10^{-4}\hat{x}_2+2.2157\times10^{-4};\\
&\varphi(\hat{x}_2)=-\frac{p\sin\alpha}{EI}\frac{\hat{x}_2^3}{6}+5.3723\times10^{-4}\frac{\hat{x}_2^2}{2}\\
&\qquad -13.9417\times10^{-4}\hat{x}_2+16.3655\times10^{-4};\\
&N(\hat{x}_2)=-p\cos\alpha\,\hat{x}_2-0.09583\times10^{-4}EA;\\
&M(\hat{x}_2)=-p\sin\alpha\frac{\hat{x}_2^2}{2}+\left(5.3723\times10^{-4}\hat{x}_2-13.9417\times10^{-4}\right)EI;\\
&T(\hat{x}_2)=-p\sin\alpha\,\hat{x}_2+5.3723\times10^{-4}EI
\end{aligned}
$$

As samples, in Figure 4, Figure 5 and Figure 6 the deflection, bending moment and shear force for the element (1) are shown, while in Figure 7, Figure 8 and Figure 9 the same quantities for the element (2) are shown.

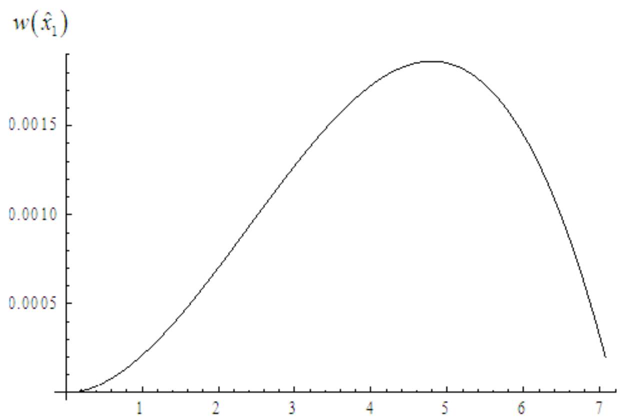

Figure 4. Deflection of the element (1) (m)

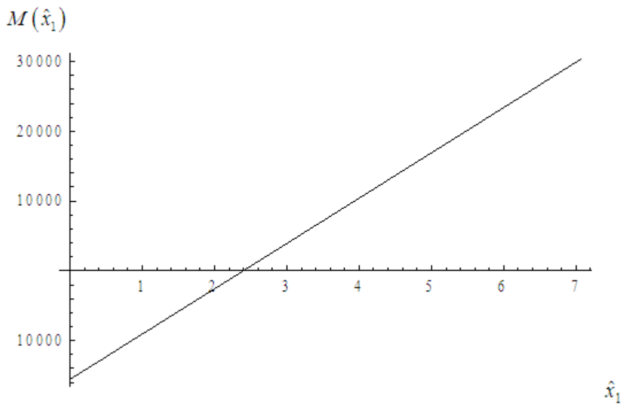

Figure 5. Bending moment of the element (1) (*Nm*).

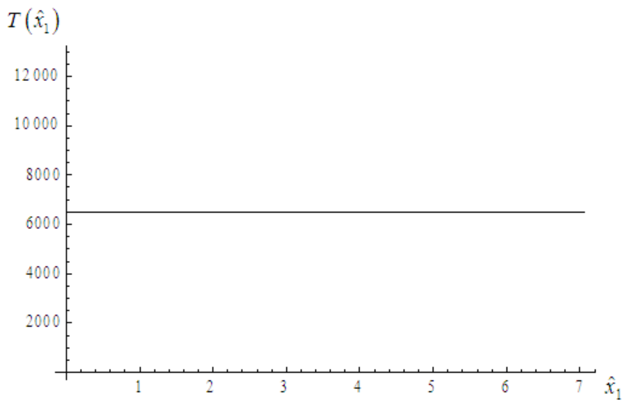

Figure 6 Shear force of the element (1) (*N*)

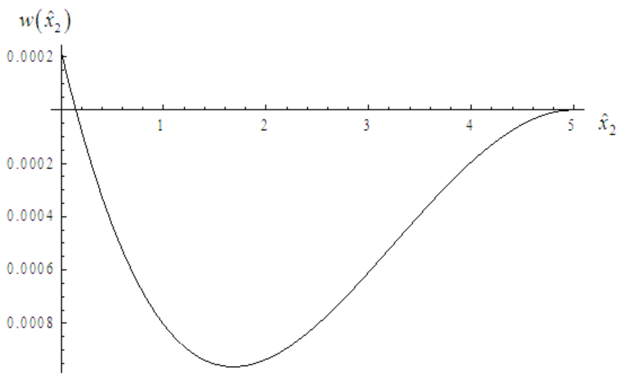

Figure 7. Deflection of the element (2) (*m*)

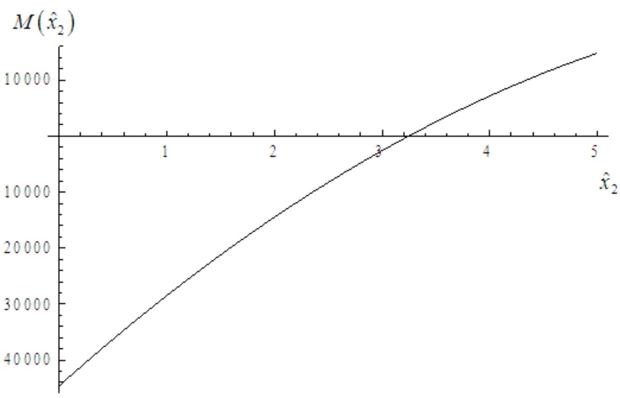

Figure 8. Bending moment of the element (2) (*Nm*)

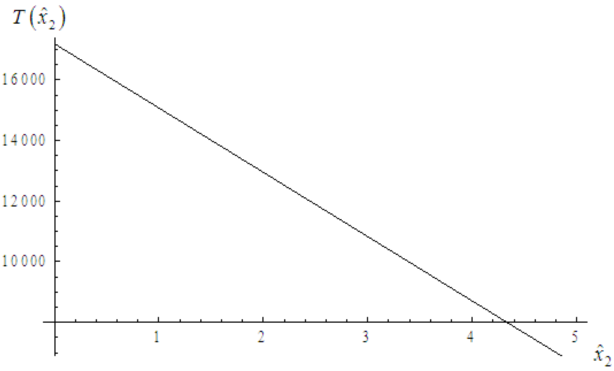

Figure 9. Shear force of the element (2) (*N*)

For the frame represented in Fig.2b the unknowns are the 30 constants inside the 5 vectors $\mathbf{c}_i$, with $i = 1,2,\ldots,5$, collected into the vector $\mathbf{c}$. The quantities defining the solving equation have the following expressions:

$$\mathbf{D} = \begin{pmatrix} \mathbf{A}_1^{(I)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_3^{(I)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_5^{(I)} \\ \mathbf{A}_1^{(J)} & -\mathbf{A}_2^{(I)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_1^{(J)} & \mathbf{B}_2^{(I)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2^{(J)} & -\mathbf{A}_3^{(J)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2^{(J)} & \mathbf{0} & -\mathbf{A}_4^{(I)} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2^{(J)} & \mathbf{B}_3^{(J)} & \mathbf{B}_4^{(I)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_4^{(J)} & -\mathbf{A}_5^{(J)} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_4^{(J)} & \mathbf{B}_5^{(J)} \end{pmatrix}; \quad (35, \text{a})$$

$$\mathbf{d} = \begin{pmatrix} \mathbf{a}_1^{(I)} \\ \mathbf{a}_3^{(I)} \\ \mathbf{a}_4^{(I)} \\ \mathbf{a}_1^{(J)} - \mathbf{a}_2^{(I)} \\ \mathbf{b}_1^{(J)} + \mathbf{b}_2^{(I)} + \mathbf{f}^{(\mathrm{A})} \\ \mathbf{a}_2^{(J)} - \mathbf{a}_3^{(J)} \\ \mathbf{a}_2^{(J)} - \mathbf{a}_4^{(I)} \\ \mathbf{b}_2^{(J)} + \mathbf{b}_3^{(J)} + \mathbf{b}_4^{(I)} \\ \mathbf{a}_4^{(J)} - \mathbf{a}_5^{(J)} \\ \mathbf{b}_4^{(J)} + \mathbf{b}_5^{(J)} + \mathbf{f}^{(\mathrm{C})} \end{pmatrix}; \quad (35, \text{b})$$

$$\mathbf{c} = \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \\ \mathbf{c}_4 \\ \mathbf{c}_5 \end{pmatrix} \quad (35, \text{c})$$

where $\mathbf{f}^{(\mathrm{A})} = \mathbf{f}^{(\mathrm{C})} = \begin{pmatrix} F & 0 & 0 \end{pmatrix}^T$.

If the following values are assumed:

$$l = 5m,\ A = 0.12m^2,\ I = 1.6\times10^{-3}m^4,$$
$$E = 2.0\times10^{10}\,N/m^2,$$
$$q = 3.0\times10^{3}\,N/m,\ F = pl$$

then the following values of the integration constants are obtained:

$$\mathbf{c}_1 = \begin{pmatrix} 0.23199 \\ 0 \\ -77.33970 \\ 150.37925 \\ 0 \\ 0 \end{pmatrix}\times10^{-5};\quad \mathbf{c}_2 = \begin{pmatrix} 0.45126 \\ 190.28927 \\ -9.23880 \\ -42.96999 \\ 134.26157 \\ 89.46906 \end{pmatrix};\quad \mathbf{c}_3 = \begin{pmatrix} -1.41746 \\ 0 \\ -9.23321 \\ 33.30093 \\ 0 \\ 0 \end{pmatrix} \tag{41a-c}$$

$$\mathbf{c}_4 = \begin{pmatrix} 0.52931 \\ 287.37825 \\ -22.61917 \\ 21.48181 \\ -9.92717 \\ 10.63095 \end{pmatrix};\quad \mathbf{c}_5 = \begin{pmatrix} -0.32341 \\ 0 \\ -7.17709 \\ 28.25471 \\ 0 \\ 0 \end{pmatrix} \tag{41d,e}$$

Starting from these values, the evaluation of any kinematic or static response of any element composing the frame is immediate.

Conclusions

A new approach for the evaluation of plane frames has been presented. It can be considered as a global, or a mixed force-displacement method because it takes into account, at the same time, the equilibrium and the compatibility conditions governing the structural mechanics of these systems. In particular, it utilizes the differential equations governing the problem of the mono-axial frames (bar and beam behavior) and considers the unknowns of the problem as the integration constants of these equations. Even if number of these unknowns is greater than the unknowns related to the use of the force method, those related to the use of the displacement method and those referring to the application of a Finite Element approach (if a not too dense discretization is used), this drawback is compensated by the fact that no post-processing is required for evaluating any static and kinematic response of the frame. The applications of the proposed approach to some simple examples has shown its feasibility and predisposition to be implemented in a computer code.

The theoretical and practical material reported in this paper may be interesting not only for engineering students, but also for any scientist working in the area of Structural Mechanics and interested to the methods for solving statically undetermined structures.

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Appendix (1)

In this appendix the vectors and the matrices appearing in Eqs (29) and (30) are given. In particular:

$$\hat{\mathbf{A}}_1^{(I)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}; \quad \hat{\mathbf{A}}_2^{(J)} = \begin{pmatrix} l & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{l^3}{6} & \frac{l^2}{2} & l & 1 \\ 0 & 0 & -\frac{l^2}{2} & -l & -1 & 0 \end{pmatrix} \tag{36, a-b}$$

while the vectors $\hat{\mathbf{a}}_1^{(I)}$ and $\hat{\mathbf{a}}_2^{(J)}$ have the following expressions

$$\hat{\mathbf{a}}_1^{(I)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T;$$

$$\hat{\mathbf{a}}_2^{(J)} = \mathbf{a}_2^{(J)} = \begin{pmatrix} \frac{l^2}{2EA} p\cos\alpha & -\frac{l^4}{24EI} p\sin\alpha & \frac{l^3}{6EI} p\sin\alpha \end{pmatrix}^T \tag{37, a-b}$$

The other matrices inside $\mathbf{D}$ are given by:

$$\mathbf{A}_1^{(J)} = \mathbf{G}_1\hat{\mathbf{A}}_1^{(J)} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2}l & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}l^3}{3} & l^2 & \sqrt{2}l & 1 \\ 0 & 0 & -l^2 & -\sqrt{2}l & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} l & \frac{\sqrt{2}}{2} & \frac{l^3}{3} & \frac{\sqrt{2}l^2}{2} & l & \frac{\sqrt{2}}{2} \\ -l & -\frac{\sqrt{2}}{2} & \frac{l^3}{3} & \frac{\sqrt{2}l^2}{2} & l & \frac{\sqrt{2}}{2} \\ 0 & 0 & -l^2 & -\sqrt{2}l & -1 & 0 \end{pmatrix} \tag{38, a}$$

$$\mathbf{A}_2^{(I)} = \hat{\mathbf{A}}_2^{(I)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{38, b}$$

$$\mathbf{B}_1^{(J)} = \mathbf{G}_1\hat{\mathbf{B}}_1^{(J)} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -EI & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}lEI & -EI & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{2}EA}{2} & 0 & -\frac{\sqrt{2}EI}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{2}EA}{2} & 0 & -\frac{\sqrt{2}EI}{2} & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2}lEI & -EI & 0 & 0 \end{pmatrix} \tag{38, c}$$

$$\mathbf{B}_2^{(I)} = \hat{\mathbf{B}}_2^{(I)} = \begin{pmatrix} -EA & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & EI & 0 & 0 & 0 \\ 0 & 0 & 0 & EI & 0 & 0 \end{pmatrix} \tag{38, d}$$

The remaining vectors to be inserted into the vector $\mathbf{d}$ are:

$$\mathbf{a}_1^{(J)} = \mathbf{G}_1\hat{\mathbf{a}}_1^{(J)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T \tag{39, a}$$

$$\mathbf{a}_2^{(I)} = \hat{\mathbf{a}}_2^{(I)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T \tag{39, b}$$

$$\mathbf{b}_1^{(J)} = \mathbf{G}_1 \hat{\mathbf{b}}_1^{(J)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T \tag{39, c}$$

$$\mathbf{b}_2^{(I)} = \hat{\mathbf{b}}_2^{(I)} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T ; \mathbf{f}^{(C)} = \begin{pmatrix} 0 & F & M \end{pmatrix}^T \tag{39, d-e}$$

Hence, the solving equations have the form:

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & l & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{l^3}{6} & \frac{l^2}{2} & l & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{l^2}{2} & -l & -1 & 0 \\
l & \frac{\sqrt{2}}{2} & \frac{l^3}{3} & \frac{\sqrt{2}l^2}{2} & l & \frac{\sqrt{2}}{2} & 0 & -1 & 0 & 0 & 0 & 0 \\
-l & -\frac{\sqrt{2}}{2} & \frac{l^3}{3} & \frac{\sqrt{2}l^2}{2} & l & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -l^2 & -\sqrt{2}l & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{\sqrt{2}EA}{2} & 0 & -\frac{\sqrt{2}EI}{2} & 0 & 0 & 0 & -EA & 0 & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{2}EA}{2} & 0 & -\frac{\sqrt{2}EI}{2} & 0 & 0 & 0 & 0 & 0 & EI & 0 & 0 & 0 \\
0 & 0 & -\sqrt{2}lEI & -EI & 0 & 0 & 0 & 0 & 0 & EI & 0 & 0
\end{pmatrix}
\begin{pmatrix} c_1^{(1)} \\ c_2^{(1)} \\ c_3^{(1)} \\ c_4^{(1)} \\ c_5^{(1)} \\ c_6^{(1)} \\ c_1^{(2)} \\ c_2^{(2)} \\ c_3^{(2)} \\ c_4^{(2)} \\ c_5^{(2)} \\ c_6^{(2)} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{l^2 p\cos\alpha}{2EA} \\ -\frac{l^4 p\sin\alpha}{24EI} \\ \frac{l^3 p\sin\alpha}{6EI} \\ 0 \\ 0 \\ 0 \\ 0 \\ F \\ M \end{pmatrix} \tag{40}$$

Substituting:

$$l = 5m,\ A = 0.12m^2,\ I = 1.6\times10^{-3}m^4,\ E = 2.0\times10^{10}\,N/m^2,\ p = 3.0\times10^3\,N/m,$$
$$F = pl,\ M = pl^2,\ \alpha = 45°$$

then the following values of the integration constants are obtained:

$$C_1^{(1)} = -0.1626\times10^{-4};\ C_2^{(1)} = 0;\ C_3^{(1)} = -2.0312\times10^{-4};\ C_4^{(1)} = 4.8669\times10^{-4};\ C_5^{(1)} = 0;$$
$$C_6^{(1)} = 0; C_1^{(2)} = -0.09583\times10^{-4};\ C_2^{(2)} = 0.5896\times10^{-4};\ C_3^{(2)} = -5.3723\times10^{-4};$$
$$C_4^{(2)} = 13.9417\times10^{-4}; C_5^{(2)} = -16.3655\times10^{-4};\ C_6^{(2)} = 2.2157\times10^{-4}$$