

Common Fixed Point of Generalized α_s - ψ -Geraghty Contractive Mappings on b -metric Spaces

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Abstract In this paper, we study the existence and uniqueness of common fixed point of generalized α_s - ψ -Geraghty contractive mapping in the framework of b -metric spaces. We give an example to prove the validity of our conclusion.

Keywords: common fixed point, weakly contractive mappings, b -metric spaces, α_s - ψ -Geraghty contractive mapping, triangular α_s , orbital admissible

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1. Introduction

The celebrated Banach Contraction Principle [1] is one of the most important fixed point results in all analysis. In 1993, the concept of b -metric space was firstly introduced by Czerwik in [2] and the author got fixed point theorems in this type space. Since then, b -metric space was studied by various researchers in different directions. For instance, Aydi et al. in [3] obtained some common fixed point results for weak φ -contractions on b -metric spaces. Berinde in [4] extended a result of usual contractions in quasimetric spaces, to a class of φ contractions. Inspired by [4], Pacurar [5] proved the existence and uniqueness of fixed point of φ -contractions. In 2018, Zada et al. [6] established fixed point results of rational contraction.

In the setting of a complete metric space, Geraghty extended the Banach contraction principle by considering an auxiliary function in [7]. In 2012, the concept of α -admissible and α - ψ -contractive mappings were introduced by Samet et al. in [8] and the authors presented some fixed point theorems for them. After that, in metric space, Cho et al. [9] introduced the concept of α -Geraghty contraction type mappings and got some fixed point results of these mappings. Recently, Özer et al. [10] established the existence and uniqueness of the common fixed point theorem for self-maps in C^* -algebra valued b -metric spaces and they obtained a result on the coupled fixed point theorems in [11]. In 2020, Özer et al. [12] proved a kind of fixed point theorem on the complete C^* -algebra valued s -metric spaces. In [13], Ullah et al. studied some strong and Δ -convergence results for mapping satisfying condition (E) in the setting of uniformly convex Busemann spaces. Kir et al. [14] established fixed point theorem for

contractive mappings satisfying contraction of Almost Jaggi type.

In metric space, by using altering distance functions, Choudhury et al. [15] studied a generalization of the weak contraction principle as follows:

Theorem 1.1. [15] Suppose that a mapping $g : L \rightarrow L$, where (L, d) is a metric space, satisfies the following condition:

$$\psi(d(ga, gb)) \leq \psi(\max\{d(a, b), d(a, ga), d(b, gb), \frac{1}{2}\{d(a, gb) + d(b, ga)\}\}) - \phi(\max\{d(a, b), d(b, gb)\}),$$

for all $a, b \in L$, where $\psi : [0, +\infty) \rightarrow [0, +\infty)$ is an altering function, that is, ψ is a nondecreasing and continuous function, and $\psi(t) = 0$ if and only if $t = 0$ and $\phi : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function. Then g has a unique fixed point.

Let (L, d) be a metric space, $T : L \rightarrow L$, and $\phi : L \rightarrow [0, +\infty)$ be a lower semicontinuous function. T is said to be a generalized weakly contractive mapping if the following condition is satisfied:

$$\psi(d(Ta, Tb) + \phi(Ta) + \phi(Tb)) \leq \psi(m(a, b, d, T, \phi)) - \phi(l(a, b, d, T, \phi)),$$

where $\psi \in \Psi, \phi \in \Phi$, and

$$m(a, b, d, T, \phi) = \max\{d(a, b) + \phi(a) + \phi(b), d(a, Ta) + \phi(a) + \phi(Ta), d(b, Tb) + \phi(b) + \phi(Tb), \frac{1}{2}\{d(a, Tb) + \phi(a) + \phi(Tb) + d(b, Ta) + \phi(b) + \phi(Ta)\}\},$$

and

$$l(a, b, d, T, \phi) = \max\{d(a, b) + \phi(a) + \phi(b), \\ d(b, Tb) + \phi(b) + \phi(Tb)\}.$$

Cho [16] generalized the results of Choudhury et al. [15] to extend weakly contractive mappings and obtained the following result:

Theorem 1.2. ([16]) Let (L, d) be complete. If T is a generalized weakly contractive mapping, then there exists a unique $r \in L$ such that $r = Tr$ and $\phi(r) = 0$.

Considering the contractive conditions which are constructed via auxiliary functions defined with the families Ψ, Φ , respectively:

$$\Psi = \{\psi : [0, +\infty) \rightarrow [0, +\infty) \text{ is a non-decreasing} \\ \text{and continuous function}\},$$

and

$$\Phi = \{\varphi : [0, +\infty) \rightarrow [0, +\infty) \text{ is a non-decreasing} \\ \text{and lower semicontinuous function and } \varphi(t) = 0 \\ \text{if and only if } t = 0\},$$

Guan [17] proved common fixed point results for a new class of generalized weakly contractive mappings.

Theorem 1.3. ([17]) Let (L, d) be a complete b -metric space with coefficient $s \geq 1$ and $f, g : L \rightarrow L$ be two given self-mappings satisfying that g is injective and $f(L) \subset g(L)$ where $g(L)$ is closed. Assume $\phi : L \rightarrow [0, +\infty)$ is a lower semicontinuous function. Let $p \geq 2$ be a constant. If there exist functions $\psi \in \Psi, \varphi \in \Phi$ such that

$$\psi(s^p [d(fa, fb) + \phi(fa) + \phi(fb)]) \leq \psi(m(a, b, d, f, g, \phi)) \\ - \varphi(l(a, b, d, f, g, \phi)),$$

where

$$m(a, b, d, f, g, \phi) = \max\{d(ga, gb) + \phi(ga) + \phi(gb), \\ \frac{1}{2}\{d(fa, ga) + \phi(fa) + \phi(ga) \\ + d(fb, gb) + \phi(fb) + \phi(gb)\}, \\ \frac{1}{2s}\{d(fa, gb) + \phi(fa) + \phi(gb) \\ + d(fb, ga) + \phi(fb) + \phi(ga)\}\},$$

and

$$l(a, b, d, f, g, \phi) = \max\{d(ga, gb) + \phi(ga) + \phi(gb), \\ d(fb, gb) + \phi(fb) + \phi(gb)\}.$$

then f and g have a unique coincidence point in L . Furthermore, they have a unique common fixed point provided that they are weakly compatible.

Throughout this paper, we aim to obtain common fixed point results for generalized α_s - ψ -Geraghty contractive mapping in the framework of b -metric space, which extended the results of Cho. Moreover, we present an example that elaborated the useability of our theorems.

2. Preliminaries

The following definitions and lemmas play the important role in obtaining our results. We state them as follows:

Definition 2.1. ([2]) Let L be a nonempty set and $s \geq 1$ be a given constant. A mapping $d : L \times L \rightarrow [0, \infty)$ is called a b -metric if and only if, for $a, b, c \in L$, the following conditions are satisfied:

- (1) $d(a, b) = 0$ if and only if $a = b$;
- (2) $d(a, b) = d(b, a)$;
- (3) $d(a, b) \leq s(d(a, c) + d(c, b))$.

In general, (L, d) is said to be a b -metric space with coefficient $s \geq 1$.

Remark 2.2. Obviously, every metric space should be a b -metric space with $s = 1$. There are many examples of b -metric spaces which are not metric spaces. (see [21])

Example 2.3. ([18]) Let (L, ρ) be a metric space, and $d(a, b) = (\rho(a, b))^p$, where $p > 1$ is a constant. Then $d(a, b)$ is a b -metric space with $s = 2^{p-1}$.

Definition 2.4. ([4]) Let (L, d) be a b -metric space with parameter $s \geq 1$. Then a sequence $\{x_n\}$ in L is said to be:

- (1) b -converges to x if and only if there exists $x \in L$ such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$;
- (2) a Cauchy sequence if and only if $d(x_n, x_m) \rightarrow 0$ when $n, m \rightarrow \infty$;

As usual, a b -metric space is called complete if and only if each Cauchy sequence in this space is b -convergent.

Definition 2.5. ([16]) Let $f, g : L \rightarrow L$ be two self-mappings. If $w = fv = gv$, for some $v \in L$, then v is called the coincidence point of f and g and w is said to be the point of coincidence of f and g . Let $C(f, g)$ denote the set of coincidence points of f and g .

Definition 2.6. ([16]) Let $f, g : L \rightarrow L$ be two self-mappings. f and g is called weakly compatible if they commute at every coincidence point.

The following lemma is important for our main results.

Lemma 2.7 ([15]) Let (L, d) be a b -metric space with parameter $s \geq 1$. We assume that $\{a_n\}$ and $\{b_n\}$ converges to a and b , respectively. Then we obtain

$$\frac{1}{s^2}d(a, b) \leq \liminf_{n \rightarrow +\infty} d(a_n, b_n) \\ \leq \limsup_{n \rightarrow +\infty} d(a_n, b_n) \\ \leq s^2d(a, b).$$

In particular, if $a = b$, then we have $\lim_{n \rightarrow \infty} d(a_n, b_n) = 0$. Moreover for each $z \in L$, we have

$$\frac{1}{s}d(a, z) \leq \liminf_{n \rightarrow +\infty} d(a_n, z) \leq \limsup_{n \rightarrow +\infty} d(a_n, z) \leq sd(a, z).$$

3. Main Results

In this part, we firstly introduce some new definitions and concepts, then we define generalized $\alpha_s - \psi -$ Geraghty contractions. Moreover, we also provide an example to support our results.

A mapping $f : L \rightarrow [0, +\infty)$ is called lower semicontinuous if, for $x \in L$ and $\{x_n\}$ is b -convergent to x , we have

$$f(x) \leq \liminf_{n \rightarrow \infty} f(x_n).$$

Let Ω denote the class of functions $\beta : \mathbb{R}_0^+ \rightarrow [0, \frac{1}{s})$ and Ψ denote the class of the functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the following conditions:

- (1) Ψ is non-decreasing,
- (2) Ψ is continuous,
- (3) $\psi(t) = 0$ iff $t = 0$.

Definition 3.1. The self-mappings $f, g : L \rightarrow L$ are said to be α_s orbital admissible and $p \geq 3$ is a constant, if the following condition holds:

$$\alpha_s(x, fx) \geq s^p, \alpha_s(x, gx) \geq s^p$$

$$\text{imply } \alpha_s(fx, gfx) \geq s^p, \alpha_s(gx, fgx) \geq s^p.$$

Definition 3.2. Let f, g be two self-mappings on L . The pair (f, g) is called triangular α_s orbital admissible and $p \geq 3$ is a constant, if

- (i) $f, g : L \rightarrow L$ are α_s orbital admissible;
- (ii) $\alpha_s(x, y) \geq s^p, \alpha_s(y, fy) \geq s^p$ and $\alpha_s(y, gy) \geq s^p$

imply $\alpha_s(x, fy) \geq s^p, \alpha_s(x, gy) \geq s^p$.

Lemma 3.3. [22] Let f, g be two self-mappings on L such that (f, g) is triangular α_s orbital admissible. Suppose that there exists $x_0 \in L$ such that $\alpha_s(x_0, fx_0) \geq s^p$. Define $\{x_n\}$ in L by $x_{2i+1} = fx_{2i}, x_{2i+2} = gx_{2i+1}$ where $i = 0, 1, 2, \dots$. The for $n, m \in \mathbb{N} \cup \{0\}$ with $m > n$, we have $\alpha_s(x_n, x_m) \geq s^p$.

Definition 3.4. Let (L, d) be a b -metric space with coefficient $s \geq 1$, and let $f, g : L \rightarrow L$ two self-mappings. Assume that $\alpha_s : L \times L \rightarrow [0, +\infty)$ and $\phi : L \rightarrow [0, +\infty)$ is a lower semicontinuous function and $p \geq 3$ is an arbitrary constant. The mappings f, g is said to be generalized $\alpha_s - \psi -$ Geraghty contractions, if there exist $\psi \in \Psi, \beta, M \geq 0$ and $\beta + M < 1, 0 < \lambda < \frac{1}{2}$ satisfying

$$\psi(\alpha_s(x, y)[d(fx, gy) + \phi(fx) + \phi(gy)]) \leq \beta\psi(r(x, y, d, f, g, \phi)) + M\psi(l(x, y, d, f, \phi)), \tag{1}$$

for all $x, y \in L$ with $\alpha_s(x, y) \geq s^p$ and $d(fx, gy) + \phi(fx) + \phi(gy) \neq 0$, where

$$r(x, y, d, f, g, \phi) = \lambda \max\{d(y, gy) + \phi(y) + \phi(gy),$$

$$d(x, y) + \phi(x) + \phi(y), d(x, fx) + \phi(x) + \phi(fx),$$

$$\frac{1}{2s}\{d(y, fx) + \phi(y) + \phi(fx) + d(x, gy) + \phi(x) + \phi(gy)\}\}$$

$$l(x, y, d, f, \phi) = \lambda \min\{d(x, fx) + \phi(x) + \phi(fx),$$

$$d(y, fx) + \phi(y) + \phi(fx), d(x, y) + \phi(x) + \phi(y)\}$$

Let (L, d) be a b -complete metric space with parameter $s \geq 1$ and $\alpha_s : L \times L \rightarrow [0, +\infty)$ be a function. Then

- (V_{s^p}) For all $x^* \in L$, one can get $\alpha_s(x^*, x^*) \geq s^p$;
- (Q_{s^p}) For all $u, v \in C(f, g)$, one can get that

$\alpha_s(u, v) \geq s^p$ or $\alpha_s(v, u) \geq s^p$.

Theorem 3.5. Let (L, d) be a complete b -metric space with coefficient $s \geq 1$ and $f, g : L \rightarrow L$ be generalized $\alpha_s - \psi -$ Geraghty contractions and f or g is continuous. If the following conditions are satisfied:

- (i) f, g are triangular α_s orbital admissible,
- (ii) there is $x_0 \in L$ with satisfying $\alpha_s(x_0, fx_0) \geq s^p$,
- (iii) properties (V_{s^p}) and (Q_{s^p}) are satisfied.

Then f and g possess a unique common fixed point. Proof. Let $x_0 \in L$. Define a sequence $\{y_n\}$ in L by $y_{2i+1} = fy_{2i}, y_{2i+2} = gy_{2i+1}$ where $i = 0, 1, 2, \dots$. Firstly, we show that f and g have at most one common fixed point. If not, there exist a and b such that $f(a) = a \neq b = g(b)$.

It follows that $d(f(a), g(b)) = d(a, b) > 0$. According to the property of (Q_{s^p}) , we have $\alpha_s(a, b) \geq s^p$, applying

(1) with $x = a$ and $y = b$, we obtain

$$\psi(d(a, b) + \phi(a) + \phi(b))$$

$$\leq \psi(s^p[d(fa, gb) + \phi(fa) + \phi(gb)]) \tag{2}$$

$$\leq \psi(\alpha_s(a, b)[d(fa, gb) + \phi(fa) + \phi(gb)])$$

$$\leq \beta\psi(r(a, b, d, f, g, \phi)) + M\psi(l(a, b, d, f, \phi)),$$

where

$$r(a, b, d, f, g, \phi) = \lambda \max\{d(b, gb) + \phi(b) + \phi(gb),$$

$$d(a, b) + \phi(a) + \phi(b), d(a, fa) + \phi(a) + \phi(fa),$$

$$\frac{1}{2s}\{d(fa, b) + \phi(fa) + \phi(b) + d(a, gb) + \phi(a) + \phi(gb)\}\}$$

$$< \frac{1}{2} \max\{d(b, b) + \phi(b) + \phi(b),$$

$$d(a, b) + \phi(a) + \phi(b), d(a, a) + \phi(a) + \phi(a),$$

$$\frac{1}{2s}\{d(a, b) + \phi(a) + \phi(b) + d(a, b) + \phi(a) + \phi(b)\}\}$$

$$\leq d(a, b) + \phi(a) + \phi(b),$$

and

$$\begin{aligned} l(a, b, d, f, \phi) &= \lambda \min \{d(a, fa) + \phi(a) + \phi(fa), \\ &\quad d(b, fa) + \phi(b) + \phi(fa), \\ &\quad d(a, b) + \phi(a) + \phi(b)\} \\ &< \frac{1}{2} \min \{d(a, a) + \phi(a) + \phi(a), \\ &\quad d(b, a) + \phi(b) + \phi(a), \\ &\quad d(a, b) + \phi(a) + \phi(b)\} \\ &\leq d(a, b) + \phi(a) + \phi(b). \end{aligned}$$

It follows from (2) that

$$\begin{aligned} &\psi(d(a, b) + \phi(a) + \phi(b)) \\ &\leq \beta \psi(d(a, b) + \phi(a) + \phi(b)) + M \psi(d(a, b) + \phi(a) + \phi(b)) \\ &< \psi(d(a, b) + \phi(a) + \phi(b)), \end{aligned}$$

which implies that $d(a, b) + \phi(a) + \phi(b) = 0$. That is, $a = b$ and $\phi(a) = 0$. Hence, the pair (f, g) has at most one common fixed point.

We suppose that $d(y_n, y_{n+1}) > 0$ for $n \in \mathbb{N}$. If not, for some k , $y_{2k} = y_{2k+1}$, by assumption (ii) and Lemma 3.3, we have $\alpha_s(y_{2k}, y_{2k+1}) \geq s^P$, and from (1), we obtain

$$\begin{aligned} &\psi(d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2})) \\ &\leq \psi(s^3 [d(fy_{2k}, gy_{2k+1}) + \phi(fy_{2k}) + \phi(gy_{2k+1})]) \\ &\leq \psi(\alpha_s(y_{2k+1}, y_{2k+2}) [d(fy_{2k}, gy_{2k+1}) \\ &\quad + \phi(fy_{2k}) + \phi(gy_{2k+1})]) \quad (3) \\ &\leq \beta \psi(r(y_{2k+1}, y_{2k+2}, d, f, g, \phi)) \\ &\quad + M \psi(l(y_{2k+1}, y_{2k+2}, d, f, \phi)), \end{aligned}$$

where

$$\begin{aligned} r(y_{2k}, y_{2k+1}, d, f, g, \phi) &= \lambda \max \{d(y_{2k+1}, gy_{2k+1}) \\ &\quad + \phi(y_{2k+1}) + \phi(gy_{2k+1}), \\ &\quad d(y_{2k}, y_{2k+1}) + \phi(y_{2k}) + \phi(y_{2k+1}), \\ &\quad d(y_{2k}, fy_{2k}) + \phi(y_{2k}) + \phi(fy_{2k}), \\ &\quad \frac{1}{2s} \{d(fy_{2k}, y_{2k+1}) + \phi(fy_{2k}) + \phi(y_{2k+1}) \\ &\quad + d(y_{2k}, gy_{2k+1}) + \phi(y_{2k}) + \phi(gy_{2k+1})\} \\ &< \frac{1}{2} \max \{d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2}), \\ &\quad d(y_{2k}, y_{2k+1}) + \phi(y_{2k}) + \phi(y_{2k+1}), \\ &\quad d(y_{2k}, y_{2k+1}) + \phi(y_{2k}) + \phi(y_{2k+1}), \\ &\quad \frac{1}{2s} \{d(y_{2k+1}, y_{2k+1}) + \phi(y_{2k+1}) + \phi(y_{2k+1}) \\ &\quad + d(y_{2k}, y_{2k+2}) + \phi(y_{2k}) + \phi(y_{2k+2})\} \\ &\leq d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2}), \end{aligned}$$

and

$$\begin{aligned} &l(y_{2k}, y_{2k+1}, d, f, \phi) \\ &= \lambda \min \{d(y_{2k}, fy_{2k}) + \phi(y_{2k}) + \phi(fy_{2k}), \\ &\quad d(y_{2k+1}, fy_{2k}) + \phi(y_{2k+1}) + \phi(fy_{2k}), \\ &\quad d(y_{2k}, y_{2k+1}) + \phi(y_{2k}) + \phi(y_{2k+1})\} \end{aligned}$$

$$\begin{aligned} &< \frac{1}{2} \min \{d(y_{2k}, y_{2k+1}) + \phi(y_{2k}) + \phi(y_{2k+1}), \\ &\quad d(y_{2k+1}, y_{2k+1}) + \phi(y_{2k+1}) + \phi(y_{2k+1}), \\ &\quad d(y_{2k}, y_{2k+1}) + \phi(y_{2k}) + \phi(y_{2k+1})\} \\ &\leq d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2}). \end{aligned}$$

By means of definition 3.4, we know that $d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2}) \neq 0$. By virtue of (3) and above inequalities, we have

$$\begin{aligned} &\psi(d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2})) \\ &\leq \beta \psi(r(y_{2k}, y_{2k+1}, d, f, g, \phi)) + L \psi(l(y_{2k}, y_{2k+1}, d, f, \phi)) \\ &\leq \beta \psi(d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2})) \\ &\quad + M \psi(d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2})) \\ &< \psi(d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2})), \end{aligned}$$

which implies that

$$d(y_{2k+1}, y_{2k+2}) + \phi(y_{2k+1}) + \phi(y_{2k+2}) = 0.$$

That is, $y_{2k+1} = y_{2k+2}$. Thus y_{2k} is a common fixed point of f and g . If $y_{2k+1} = y_{2k+2}$, then the proof is too similar to the case $y_{2k} = y_{2k+1}$, one can show that y_{2k+1} is a common fixed point of f and g .

Now take $d(y_n, y_{n+1}) > 0$ for each $n \in \mathbb{N}$. Letting $x = y_{2n}$ and $y = y_{2n+1}$ in (1), as the same arguments, we obtain

$$\begin{aligned} &\psi(d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2})) \\ &\leq \beta \psi(r(y_{2n}, y_{2n+1}, d, f, g, \phi)) \quad (4) \\ &\quad + M \psi(l(y_{2n}, y_{2n+1}, d, f, \phi)), \end{aligned}$$

where

$$\begin{aligned} r(y_{2n}, y_{2n+1}, d, f, g, \phi) &= \lambda \max \{d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2}), \\ &\quad d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1}), \\ &\quad d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1}), \\ &\quad \frac{1}{2s} \{d(y_{2n+1}, y_{2n+1}) + \phi(y_{2n+1}) + \phi(y_{2n+1}) \\ &\quad + d(y_{2n}, y_{2n+2}) + \phi(y_{2n}) + \phi(y_{2n+2})\} \\ &\leq \frac{1}{2} \max \{d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1}), \\ &\quad d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2})\}, \end{aligned} \quad (5)$$

and

$$\begin{aligned} &l(y_{2n}, y_{2n+1}, d, f, \phi) \\ &= \lambda \min \{d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1}), \\ &\quad d(y_{2n+1}, y_{2n+1}) + \phi(y_{2n+1}) + \phi(y_{2n+1}), \\ &\quad d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1})\} \\ &\leq d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1}). \end{aligned} \quad (6)$$

If for some n ,

$$\begin{aligned} &d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2}) \\ &> d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1}), \end{aligned}$$

then it follows from (4), (5) and (6) that

$$\begin{aligned} &\psi(d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2})) \\ &\leq \beta\psi(d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2})) \\ &+ M\psi(d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2})) \\ &< \psi(d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2})), \end{aligned}$$

which yields that

$$d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2}) = 0.$$

That is, $d(y_{2n+1}, y_{2n+2}) = 0$, a contradiction. Therefore,

$$\begin{aligned} &d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2}) \\ &\leq d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1}), \end{aligned} \tag{7}$$

for all $n \in \mathbb{N}$. Using similar arguments, we get

$$\begin{aligned} &d(y_{2n+2}, y_{2n+3}) + \phi(y_{2n+2}) + \phi(y_{2n+3}) \\ &\leq d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2}). \end{aligned}$$

Therefore, $\{d(y_n, y_{n+1}) + \phi(y_n) + \phi(y_{n+1})\}$ is a non-increasing sequence and there exists a $r \geq 0$ such that

$$\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) + \phi(y_n) + \phi(y_{n+1}) = r.$$

If $r > 0$, by virtue of (4), (5), (6) and (7), one can obtain that

$$\begin{aligned} &\psi(d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2})) \\ &\leq \beta\psi(d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1})) \\ &+ M\psi(d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1})). \end{aligned} \tag{8}$$

Taking the limit as $n \rightarrow +\infty$ in (8), we get

$$\begin{aligned} \psi(r) &= \lim_{n \rightarrow +\infty} \psi(d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2})) \\ &\leq \beta \lim_{n \rightarrow +\infty} \psi(d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1})) \\ &+ M \lim_{n \rightarrow +\infty} \psi(d(y_{2n}, y_{2n+1}) + \phi(y_{2n}) + \phi(y_{2n+1})) \\ &< \lim_{n \rightarrow +\infty} \psi(d(y_{2n+1}, y_{2n+2}) + \phi(y_{2n+1}) + \phi(y_{2n+2})) \\ &= \psi(r), \end{aligned}$$

which gives a contradiction. It follows that

$$\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) + \phi(y_n) + \phi(y_{n+1}) = 0.$$

and which yields that

$$\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) = 0, \lim_{n \rightarrow +\infty} \phi(y_n) = 0.$$

Next, we shall prove that $\{y_n\}$ is a Cauchy sequence in L . Obviously, it is sufficient to show that $\{y_{2n}\}$ is Cauchy. Assume that $\{y_{2n}\}$ is not Cauchy. It follows that there exists $\varepsilon > 0$ for which one can choose $\{y_{2m_k}\}$ and $\{y_{2n_k}\}$ of $\{y_{2n}\}$ satisfying n_k is the smallest index for which $2m_k > 2n_k > k$,

$$\begin{aligned} &\varepsilon \leq d(y_{2m_k}, y_{2n_k}), \\ &d(y_{2m_k-2}, y_{2n_k}) < \varepsilon. \end{aligned}$$

By the triangle inequality in b -metric space, we can deduce that

$$\varepsilon \leq \limsup_{k \rightarrow +\infty} d(y_{2m_k}, y_{2n_k}) \leq s\varepsilon, \tag{9}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k \rightarrow +\infty} d(y_{2m_k-1}, y_{2n_k}) \leq s^2\varepsilon, \tag{10}$$

$$\frac{\varepsilon}{s^2} \leq \limsup_{k \rightarrow +\infty} d(y_{2m_k-1}, y_{2n_k+1}) \leq s^3\varepsilon, \tag{11}$$

$$\frac{\varepsilon}{s} \leq \limsup_{k \rightarrow +\infty} d(y_{2m_k}, y_{2n_k+1}) \leq s^2\varepsilon. \tag{12}$$

Letting $x = y_{2n_k}$ and $y = y_{2m_k-1}$ in (1), by Lemma 3.3, we know that $\alpha_s(y_{2n_k}, y_{2m_k-1}) \geq s^P$, so we obtain

$$\begin{aligned} &\psi(d(y_{2n_k+1}, y_{2m_k}) + \phi(y_{2n_k+1}) + \phi(y_{2m_k})) \\ &\leq \psi(s^P [d(fy_{2n_k}, gy_{2m_k-1}) + \phi(fy_{2n_k}) + \phi(gy_{2m_k-1})]) \\ &\leq \psi(\alpha_s(y_{2n_k}, y_{2m_k-1}) \left[\begin{aligned} &d(fy_{2n_k}, gy_{2m_k-1}) \\ &+ \phi(fy_{2n_k}) + \phi(gy_{2m_k-1}) \end{aligned} \right]) \tag{13} \\ &\leq \beta\psi(r(y_{2n_k}, y_{2m_k-1}, d, f, g, \phi)) \\ &+ M\psi(l(y_{2n_k}, y_{2m_k-1}, d, f, \phi)). \end{aligned}$$

Here,

$$\begin{aligned} &r(y_{2n_k}, y_{2m_k-1}, d, f, g, \phi) \\ &= \lambda \max\{d(y_{2m_k-1}, gy_{2m_k-1}) + \phi(y_{2m_k-1}) + \phi(gy_{2m_k-1}), \\ &d(y_{2n_k}, y_{2m_k-1}) + \phi(y_{2n_k}) + \phi(y_{2m_k-1}), \\ &d(y_{2n_k}, fy_{2n_k}) + \phi(y_{2n_k}) + \phi(fy_{2n_k}), \\ &\frac{1}{2s}\{d(fy_{2n_k}, y_{2m_k-1}) + \phi(fy_{2n_k}) + \phi(y_{2m_k-1}) \\ &+ d(y_{2n_k}, gy_{2m_k-1}) + \phi(y_{2n_k}) + \phi(gy_{2m_k-1})\} \\ &= \lambda \max\{d(y_{2m_k-1}, y_{2m_k}) + \phi(y_{2m_k-1}) + \phi(y_{2m_k}), \\ &d(y_{2n_k}, y_{2m_k-1}) + \phi(y_{2n_k}) + \phi(y_{2m_k-1}), \\ &d(y_{2n_k}, y_{2n_k+1}) + \phi(y_{2n_k}) + \phi(y_{2n_k+1}), \\ &\frac{1}{2s}\{d(y_{2n_k+1}, y_{2m_k-1}) + \phi(y_{2n_k+1}) + \phi(y_{2m_k-1}) \\ &+ d(y_{2n_k}, y_{2m_k}) + \phi(y_{2n_k}) + \phi(y_{2m_k})\} \}, \end{aligned}$$

and

$$\begin{aligned} &l(y_{2n_k}, y_{2m_k-1}, d, f, \phi) \\ &= \lambda \min\{d(y_{2n_k}, fy_{2n_k}) + \phi(y_{2n_k}) + \phi(fy_{2n_k}), \\ &d(y_{2m_k-1}, fy_{2n_k}) + \phi(y_{2m_k-1}) + \phi(fy_{2n_k}), \\ &d(y_{2n_k}, y_{2m_k-1}) + \phi(y_{2n_k}) + \phi(y_{2m_k-1})\} \\ &= \lambda \min\{d(y_{2n_k}, y_{2n_k+1}) + \phi(y_{2n_k}) + \phi(y_{2n_k+1}), \\ &d(y_{2m_k-1}, y_{2n_k+1}) + \phi(y_{2m_k-1}) + \phi(y_{2n_k+1}), \\ &d(y_{2n_k}, y_{2m_k-1}) + \phi(y_{2n_k}) + \phi(y_{2m_k-1})\}. \end{aligned}$$

It follows from (9)-(12) that

$$\begin{aligned} & \limsup_{k \rightarrow +\infty} r(y_{2n_k}, y_{2m_k-1}, d, f, g, \phi) \\ & \leq \lambda \max\{0, s^2 \varepsilon, 0, \frac{s^3 \varepsilon + s \varepsilon}{2s}\} \\ & \leq s^2 \varepsilon, \end{aligned} \tag{14}$$

and

$$\begin{aligned} & \limsup_{k \rightarrow +\infty} l(y_{2n_k}, y_{2m_k-1}, d, f, \phi) \\ & \leq \lambda \min\{0, s^3 \varepsilon, s^2 \varepsilon\} \\ & \leq s^2 \varepsilon, \end{aligned} \tag{15}$$

By virtue of (13), (14) and (15), we have

$$\begin{aligned} \psi(s^2 \varepsilon) &= \psi(s^3 \frac{\varepsilon}{s}) \leq \psi(s^p \frac{\varepsilon}{s}) \\ &\leq \psi(s^p \limsup_{k \rightarrow +\infty} [d(y_{2n_k+1}, y_{2m_k}) + \phi(y_{2n_k+1}) + \phi(y_{2m_k})]) \\ &\leq \psi(\limsup_{k \rightarrow +\infty} \alpha_s(y_{2n_k}, y_{2m_k-1}) [d(y_{2n_k+1}, y_{2m_k}) \\ &+ \phi(y_{2n_k+1}) + \phi(y_{2m_k})]) \\ &\leq \beta \psi(s^2 \varepsilon) + M \psi(s^2 \varepsilon) \\ &< \psi(s^2 \varepsilon), \end{aligned}$$

which is a contradiction. Hence, $\{y_n\}$ is Cauchy. It follows from the completeness of L that there exists a z^* in L such that

$$\lim_{n \rightarrow +\infty} f y_{2n} = \lim_{n \rightarrow +\infty} g y_{2n+1} = z^*. \tag{16}$$

Considering the definition of ϕ , we deduce that

$$\phi(x^*) \leq \liminf_{n \rightarrow +\infty} \phi(y_n) = 0.$$

Next we shall prove that if one of the mappings f and g is continuous, then $fz^* = gz^* = z^*$. Without loss of generality, one can assume that f is a continuous mapping. From (16), one can deduce that

$$z^* = \lim_{n \rightarrow +\infty} f y_{2n} = f(\lim_{n \rightarrow +\infty} y_{2n}) = f(z^*).$$

That is, z^* is a fixed point of f .

Using the property of (V_{s^p}) , we obtain $\alpha_s(z^*, z^*) \geq s^p$.

If $z^* \neq gz^*$, by the contractive conditions (1), we get

$$\begin{aligned} & \psi(d(z^*, gz^*) + \phi(z^*) + \phi(gz^*)) \\ & \leq \psi(s^p [d(fz^*, gz^*) + \phi(fz^*) + \phi(gz^*)]) \\ & \leq \psi(\alpha_s(z^*, z^*) [d(fz^*, gz^*) + \phi(z^*) + \phi(gz^*)]) \\ & \leq \beta \psi(r(z^*, z^*, d, f, g, \phi)) + M \psi(l(z^*, z^*, d, f, \phi)), \end{aligned} \tag{17}$$

where

$$\begin{aligned} & (r(z^*, z^*, d, f, g, \phi) = \lambda \max\{d(z^*, gz^*) + \phi(z^*) + \phi(gz^*), \\ & d(z^*, z^*) + \phi(z^*) + \phi(z^*), d(z^*, fz^*) + \phi(z^*) + \phi(fz^*), \\ & \frac{1}{2s} \{d(fz^*, z^*) + \phi(fz^*) + \phi(z^*) \\ & + d(z^*, gz^*) + \phi(z^*) + \phi(gz^*)\}\}) \\ & = \lambda \max\{d(z^*, gz^*) + \phi(gz^*), 0, 0, \\ & \frac{1}{2s} \{d(z^*, gz^*) + \phi(gz^*)\}\} \\ & \leq d(z^*, gz^*) + \phi(gz^*), \end{aligned}$$

and

$$l(z^*, z^*, d, f, \phi) = 0 \leq d(z^*, gz^*) + \phi(gz^*).$$

It follows from (17) that

$$\begin{aligned} & \psi(d(z^*, gz^*) + \phi(z^*) + \phi(gz^*)) \leq \beta \psi(d(z^*, gz^*) + \phi(gz^*)) \\ & + M \psi(d(z^*, gz^*) + \phi(gz^*)) \\ & < \psi(d(z^*, gz^*) + \phi(gz^*)). \end{aligned}$$

Hence, $d(z^*, gz^*) + \phi(gz^*) = 0$, that is, $z^* = gz^*$ and $\phi(gz^*) = 0$, a contradiction. This implies that z^* is the unique common fixed point of f and g . This completes the proof.

Example 3.6. Let $L = [0, +\infty)$ and $d(x, y) = (x - y)^2$ for $x, y \in L$. Define mappings $f, g : L \rightarrow L$ by

$$fx = \begin{cases} \frac{x}{64}, & x \in [0, 1] \\ e^x, & x > 1, \end{cases}$$

and

$$gx = \begin{cases} \frac{x}{16}, & x \in [0, 1] \\ e^{2x} - e^2 + \frac{1}{16}, & x > 1. \end{cases}$$

Put $\alpha_s : L \times L \rightarrow [0, +\infty)$ by

$$\alpha_s(x, y) = \begin{cases} s^3, & x, y \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Define mappings $\psi : [0, +\infty) \rightarrow [0, +\infty)$ and $\phi : L \rightarrow L$

with $\psi(t) = 2t, \varphi(x) = \frac{x^2}{4}$. Let $\beta = \frac{9}{10}, M = \frac{1}{100}$ and $\lambda = \frac{1}{3}$.

For $x, y \in L$ such that $\alpha_s(x, y) \geq s^3$, we deduce that $x, y \in [0, 1]$. So we have $\alpha_s(x, fx) \geq s^3, \alpha_s(x, gx) \geq s^3$ imply $\alpha_s(fx, gfx) \geq s^3, \alpha_s(gx, fgx) \geq s^3$, that is, f, g are α_s orbital admissible.

$$\begin{aligned} &\psi(\alpha_s(x, y)[d(fx, gy) + \phi(fx) + \phi(gy)]) \\ &= 2^4 \cdot [(\frac{x}{64} - \frac{y}{16})^2 + \frac{1}{4}(\frac{x}{64})^2 + \frac{1}{4}(\frac{y}{16})^2] \\ &= 16 \cdot (\frac{x}{64} - \frac{y}{16})^2 + 4(\frac{x}{64})^2 + 4(\frac{y}{16})^2 \\ &\leq 20 \cdot (\frac{x}{64})^2 + 20 \cdot (\frac{y}{16})^2 \\ &= \frac{20}{64^2}x^2 + \frac{20}{16^2}y^2, \\ &\beta\psi(r(x, y, d, f, g, \phi)) \\ &\geq \frac{9}{10}[\psi(\frac{1}{3} \cdot \frac{1}{2}(d(fx, x) + \phi(fx) + \phi(x) \\ &\quad + d(y, gy) + \phi(y) + \phi(gy)))] \\ &= \frac{3}{10} \left[\begin{aligned} &(\frac{x}{64} - x)^2 + \frac{1}{4}(\frac{x}{64})^2 + \frac{x^2}{4} \\ &+ (\frac{y}{16} - y)^2 + \frac{1}{4}(\frac{y}{16})^2 + \frac{y^2}{4} \end{aligned} \right] \\ &= \frac{3}{10}(\frac{19973}{16384}x^2 + \frac{1157}{1024}y^2) \\ &= \frac{59919}{163840}x^2 + \frac{3471}{10240}y^2, \end{aligned}$$

According to above inequalities, it suffices to verify that

$$\begin{aligned} &\psi(\alpha_s(x, y)[d(fx, gy) + \phi(fx) + \phi(gy)]) \\ &\leq \frac{20}{64^2}x^2 + \frac{20}{16^2}y^2 \\ &\leq \frac{59919}{163840}x^2 + \frac{3471}{10240}y^2 \\ &\leq \beta\psi(r(x, y, d, f, g, \phi)) \\ &\quad + M\psi(l(x, y, d, f, \phi)). \end{aligned}$$

It is easy to show that all conditions of Theorem 3.5 are satisfied with $s = 2$. Obviously, 0 is the unique common fixed point of f and g .

If $\phi = 0$ in Theorem 3.5, we obtain immediately Corollary 3.7:

Corollary 3.7. Let (L, d) be a complete b - metric space with $s \geq 1$ and $f, g : L \rightarrow L$ be given two self-mappings and one of f and g is continuous. If the following conditions are fulfilled:

- (i) f, g are triangular α_s orbital admissible,
- (ii) there is $x_0 \in L$ with satisfying $\alpha_s(x_0, fx_0) \geq s^p$,
- (iii) if there are $\psi \in \Psi, \beta, M \geq 0$ and

$$\beta + M < 1, 0 < \lambda < \frac{1}{2} \text{ satisfying}$$

$$\psi(\alpha_s(a, b)d(fa, gb)) \leq \beta\psi(r_1(a, b)) + M\psi(l_1(a, b)),$$

for all $a, b \in L$ with $\alpha_s(a, b) \geq s^p$ and $d(fa, gb) \neq 0$, where

$$\begin{aligned} r_1(a, b) &= \lambda \max\{d(b, gb), d(a, b), d(a, fa), \\ &\frac{1}{2s}\{d(b, fa) + d(a, gb)\}\}, \end{aligned}$$

$$l_1(a, b) = \lambda \min\{d(a, fa), d(b, fa), d(a, b)\}.$$

(iv) properties (V_{s^p}) and (Q_{s^p}) are satisfied.

Then f and g possess a unique common fixed point.

If we consider $M = 0$ in Theorem 3.5, we get that

Corollary 3.8. Let (L, d) be a complete b - metric space with $s \geq 1$ and let $f, g : L \rightarrow L$ be two given self-mappings and one of f and g is continuous. If the following conditions are satisfied:

- (i) f, g are triangular α_s orbital admissible,
- (ii) there is $x_0 \in L$ with satisfying $\alpha_s(x_0, fx_0) \geq s^p$,
- (iii) if there are $\psi \in \Psi$ and $\beta < 1, 0 < \lambda < \frac{1}{2}$ satisfying

$$\begin{aligned} &\psi(\alpha_s(x, y)[d(fx, gy) + \phi(fx) + \phi(gy)]) \\ &\leq \beta\psi(r(x, y, d, f, g, \phi)), \end{aligned}$$

for $x, y \in L$ such that $\alpha_s(x, y) \geq s^p$ and $d(fx, gy) \neq 0$,

(iv) properties (V_{s^p}) and (Q_{s^p}) are satisfied.

Then f and g possess a unique common fixed point.

Remark 3.9. If $\phi = 0$ and $s = 1$ in Corollary 3.8, taking $S = T = I_x$ in Theorem 2.1 of [20], Roshan et al. established the existence theorem of common fixed point for mappings f, g satisfying

$$\begin{aligned} d(fx, gy) &\leq \frac{q}{s^4} \max\{d(x, y), d(fx, x), \\ &d(gy, y), \frac{1}{2}(d(x, fy) + d(fx, y))\}, \end{aligned}$$

where $q \in (0, 1)$ is a constant. For $x = 0, y \in (\frac{1}{2}, 1]$, it is

easy to calculate that for $q < \frac{1}{16}$,

$$\begin{aligned} d(fx, gy) &= \frac{y^2}{256} > \frac{q}{16}y^2 \\ &= \frac{q}{16} \max\{y^2, 0, \frac{225y^2}{256}, \frac{y^2}{2 \cdot 256} + \frac{y^2}{2}\} \\ &= \frac{q}{s^4} \max\{d(x, y), d(fx, x), d(gy, y), \\ &\frac{1}{2}(d(x, fy) + d(fx, y))\}. \end{aligned}$$

One can easily to obtain that Theorem 2.1 of [20] can not be applied to get the existence of common fixed points of the mappings f and g in L .

4. Application

In this section, we wish to study the existence of a solution for a pair of boundary value problems. Let $C[0, 1]$ denote the space of all continuous function defined on $[0, 1]$. Consider the following differential equations:

$$\begin{aligned}
-\frac{d^2x}{dt^2} &= F(t, x(t)), \quad t \in [0,1] \\
x(0) &= x(1) = 0. \\
\text{and} & \\
-\frac{d^2y}{dt^2} &= K(t, y(t)), \quad t \in [0,1] \\
y(0) &= y(1) = 0.
\end{aligned} \tag{18}$$

where $F, K : [0,1] \times C[0,1] \rightarrow \mathbb{R}$ are continuous functions. Associated with (18), the Green function is defined by

$$G(t, s) = \begin{cases} t(1-\tau) & 0 \leq t \leq \tau \leq 1 \\ \tau(1-t) & 0 \leq \tau \leq t \leq 1. \end{cases}$$

Define $d : C[0,1] \times C[0,1] \rightarrow [0, +\infty)$ by $d(x, y) = \sup_{t \in I} |x(t) - y(t)|^p$, for $x, y \in C[0,1]$. It is easy to show that $(C[0,1], d)$ is a complete b -metric space with coefficient $s = 2^{p-1}$. We define the operators $f, g : C[0,1] \rightarrow C[0,1]$ by

$$fx(t) = \int_0^1 G(t, r) F(\tau, x(\tau)) dr$$

and

$$gx(t) = \int_0^1 G(t, r) K(\tau, x(\tau)) dr$$

for all $t \in [0,1]$ and let $\xi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a given function.

Theorem 4.1. Suppose that

- (i) $F, K : [0,1] \times C[0,1] \rightarrow \mathbb{R}$ are continuous,
- (ii) there is $x_0 \in C[0,1]$ with satisfying $\xi(x_0, fx_0) \geq 0$ for all $t \in [0,1]$,
- (iii) For all $t \in [0,1]$ and $x, y \in C[0,1]$,

$$\xi(x(t), fx(t)) \geq s^p, \xi(x(t), gx(t)) \geq s^p$$

imply $\xi(fx(t), gfx(t)) \geq s^p, \xi(gx(t), fgx(t)) \geq s^p$,

$$\xi(x(t), y(t)) \geq s^p, \xi(y(t), fy(t)) \geq s^p$$

and

$$\xi(y(t), gy(t)) \geq s^p$$

imply $\xi(x(t), fy(t)) \geq s^p, \xi(x(t), gy(t)) \geq s^p$,

(iv) properties $(V_{s,p})$ and $(Q_{s,p})$ are satisfied,

(v) For all $x, y \in C[0,1]$, $t \in [0,1]$,

$$|F(t, x) - K(t, x)| \leq p \sqrt[p]{\frac{1}{s^{p+1}}} |x(t) - y(t)|.$$

Then (18) have a unique solution $x \in C[0,1]$.

Proof. Define $\alpha_s : C[0,1] \times C[0,1] \rightarrow [0, +\infty)$ by

$$\alpha_s(x, y) = \begin{cases} s^p & \text{if (iii) holds,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to prove that f, g are triangular α_s orbital admissible. For $x, y \in C(I)$, by virtue of assumptions (i)-(v), we have

$$\begin{aligned}
s^p d(fx(t), gy(t)) &= s^p \sup_{t \in I} |fx(t) - gy(t)|^p \\
&= s^p \sup_{t \in I} \left| \int_0^1 G(t, \tau) [F(\tau, x(\tau)) - K(\tau, y(\tau))] d\tau \right|^p \\
&\leq s^p \sup_{t \in I} \left| \int_0^1 G(t, \tau) \sqrt[p]{\frac{1}{s^{p+1}}} |x(\tau) - y(\tau)| d\tau \right|^p \\
&\leq s^p \sup_{t \in I} \left(\int_0^1 G(t, \tau) \sqrt[p]{\frac{1}{s^{p+1}}} d\tau \right)^p \sup_{t \in I} \|x(t) - y(t)\|^p \\
&\leq \frac{1}{s} \psi(r(x, y, d, f, g, \phi))
\end{aligned}$$

which implies that

$$\begin{aligned}
\psi(\alpha_s(x, y)[d(fx, gy) + \phi(fx) + \phi(gy)]) \\
\leq \beta \psi(r(x, y, d, f, g, \phi)),
\end{aligned}$$

Therefore, letting $\psi(t) = t$, $\beta = \frac{1}{s}$, and $\phi(t) = 0$, all the conditions of Corollary 3.8 are satisfied. As a result, the mapping f and g have a unique fixed point $x \in C[0,1]$, which is a solution of (18).

5. Conclusions

In this manuscript, we introduced a new class of generalized Geraghty contractive mapping and established common fixed point results involving this new class of mappings in the framework of b -metric spaces. Furthermore, we presented examples that elaborated the useability of our results. Meanwhile, we provided an application to the existence of a solution for a pair of boundary value problems by means of one of our results.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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