

Solution of a System of HIV Model Equations by the Variational Iteration Method

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Abstract Mathematical modeling of many biological systems leads to ordinary differential equations (ODEs), which are often too complicated to solve exactly. Acquire Immune Deficiency Syndrome (AIDS) is one of the greatest health challenges of this millennium and it is caused by a virus called Human Immunodeficiency Virus (HIV). This work is a nonlinear mathematical model of HIV/AIDS dynamics considering Counseling and Anti-Retroviral Therapy (ART) which was developed in the form of differential equation. Three sub-models of the general model considered were the sub-model without ART, the sub-model with only infected males receiving ART and the sub-model with only infected females receiving ART. The general model and the sub-models with various parameter values are solved using the Variational Iteration Method (VIM), which is a semi analytical method. The VIM is used to obtain solutions of both nonlinear and linear functional equations without discretizing the equations or approximating the operators. The solution when it exists is found in a rapidly converging series form. The VIM provided continuous solutions to the model which can be used for further analysis like differentiation and integration and can be used to compute prevalence rates. Solutions of the model, presented in graphical form and the results revealed that VIM is an alternative method for the fourth-order Runge Kutta method. It was also observed that for effective counseling and ART to lead eradication, it necessary that the same proportion of males and females should be involved in ART. The existence of the disease free equilibrium state of the general model is investigated and shown to be locally and asymptotically stable (LAS).

Keywords: virus, susceptible, infective, counseling, variation, eradication

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1. Introduction

Mathematical modeling of many biological or physical systems leads to ordinary differential equations. Ordinary differential equations which arise in real-world physical problems are often too complicated to solve exactly. Even if an exact solution is obtainable, the required calculations may be too complicated to be practical, or the resulting solution may be difficult to interpret. In recent years, there has been interest in semi-analytic method for solving nonlinear functional equations such as Adomian decomposition method (ADM), Homotopy perturbation method (HDM) etc. [1]. In this work, we are particularly concerned with Variational Iteration Method (VIM) which was developed by Ji-Huan He in 1999 as a tool for solving Differential Equations (DEs). The method is now widely used by many researchers to study linear and nonlinear differential equations. The method gives rapidly convergent series approximations of the exact solution if such a solution exists; otherwise a few approximations can

be used for numerical purposes [2]. The variational iteration method has no specific requirements such as linearization, small parameters and any transformation for nonlinear operators. It is the goal of this work to effectively employ the variational iteration method to compute approximate solutions of a two-sex mathematical model for HIV/AIDS considering counseling and ART developed by Kimbir *et al.* (2012) [3].

This model extends the one-sex mathematical model for the transmission dynamics of HIV/AIDS considering counseling and ART by Kimbir and Oduwole (2008) [4]. The model is a system of nonlinear ordinary differential equations. To this end, this research work seeks to solve a nonlinear HIV model of Kimbir *et al.* (2012) using variational iteration method to obtain reliable and accurate approximations in a series form.

Soltani and Shizadi (2010) presented a new modification of the variational iteration method, which provides great freedom in choosing linear operator for various nonlinear equations, so that the Lagrange multiplier can be effectively identified [1]. Hemeda (2009) used the variational iteration method to solve nonlinear partial

differential equations such as the combined KdV-KmdV equation and Camassa-Holm equation [6]. Saadatmandi and Dehghan (2009) applied the variational iteration method to the wave-equation subject to an integral conservation condition which yields exact solution of the problem [7]. Goh *et al.* (2010) applied this method to solve a biochemical reaction model. This model is a system of nonlinear ordinary differential equations. Due to the nature of the model, the result from variational iteration method was not in order [8]. They modified variational iteration method, which they called Multistage Variational Iteration Method (MVIM) and applied the modified method to the model. The MVIM obtained a better result than the standard variational iteration method for this model.

Fernandez (2008) used the variational iteration method to solve a prey-predator model with variable coefficients and obtained increasingly accurate approximation solutions which do not exhibit the pole of the exact answer, and tends to zero as t tends to infinity [9]. Aminikhah (2012) obtains the solution of wave-equation in radial form by variational iteration method [10]. Liu (2012) applied the variational iteration method to strongly nonlinear q -difference equations. The result reveals that the method is very effective and can be applied to other nonlinear q -difference equations [11]. Hammouch and Mekkaoui (2012) used a Laplace variational iteration method for solving the homogenous Smoluchowski coagulation equation [12]. Noor and Mohyud-Din (2008) developed a Variational Iteration Decomposition Method (VIDM) and used it in solving higher dimensional initial boundary value problems by using the equivalent system of integral equations [13].

Jassim (2012) modified the variational iteration method for solving Schrodinger and Laplace problems. This modified method choose linear operator for various nonlinear equations so that the Lagrange multiplier can be effectively identified [14]. Noor and Mahyud-Din (2008) used the method to solve initial and boundary value problems of Bratu-type which are widely applicable in fuel ignition of the combustion theory and heat transfer [13,14]. The result clearly indicates the reliability and the accuracy of method. They observed the fact that the method solves nonlinear problems without using Adomian polynomials can be considered as a clear advantage of this method over the Adomian decomposition method. Mirzaei (2011) applied homotopy perturbation method and variational iteration method for Volterra integral equations. The result reveals that the homotopy perturbation method and variational iteration method are effective [1].

Thus we see from this review that the variational iteration method has been used to solve many linear and nonlinear functional equations and the results obtained affirm that the method is a reliable one for continuous solutions and it does not require discretization.

Several physical and biological problems can be modelled using systems of nonlinear ordinary differential equation. For example, Capasso (2008) in his book provides a detailed exposition on the mathematical structures of epidemic systems [15]. He discussed several methods of analysis including the substitution method, linearization method, Liapunov's second method,

Routh-Hurwitz criterion, Hopf-bifurcation theorem etc. Benyah (2007) presented a paper on epidemiological modeling and analysis. He used substitution method to obtain a critical point, Linearization at equilibrium points, Variable separable etc [16].

Naresh and Tripsthi (2005) proposed a nonlinear mathematical model for the transmission dynamics of fly and curable TB pathogens within a population of a varying size. The mode consist of four sub-classes of susceptible, TB infective, HIV infective and that of AIDS patients. Based on their study, it was found that if the migration of the population into susceptible is restricted; the spread of the disease can be kept under control [17]. The number of HIV infected individuals in the population increases rapidly due to the presence of tuberculosis. The model analysis suggests that if the HIV infection is suppressed at early stage by way of drug therapy or other control mechanisms, the progression to AIDS can be slowed down and the life span HIV infectives can be increased [17].

Naresh *et al.* (2011) proposed a nonlinear HIV/AIDS model and analyzed it to study the effect of contact tracing on reducing the spread of HIV/AIDS in a homogenous population with constant immigration of susceptibles. In modeling the dynamics, the population is divided into four subclasses of HIV negatives but susceptibles, HIV positive or infectives that do not know they are infected, HIV positive that know they are infected and that of AIDS patients. Susceptibles are assumed to become infected via sexual contacts with (both types of) infectives and all infectives move with constant rates to develop AIDS [18]. The model is analyzed using the stability theory of differential equations and numerical simulations. The model analysis shows that the spread of AIDS epidemic in a population. It is also found that the endemicity of infection is reduced when infectives after becoming aware of their infection do not take part in sexual interaction [15].

de Arazoza and Lounes (2002) developed a nonlinear model for an epidemic with contact tracing, and studied its dynamic. They presented the data for the Cuban HIV/AIDS epidemic and fit the nonlinear model, the obtained estimates for the size of the Cuban HIV epidemic, and for the mean time for detecting a person that is infected with HIV [19].

Kimbir and Oduwole (2008) developed and studied a one-sex mathematical model on HIV transmission dynamics considering Counseling and Antiretroviral Therapy, ART. Their study showed that counseling and ART are good strategies for the control and eradication of HIV/AIDS. Kimbir *et al.* (2012) formulated and studied a two-sex mathematical model for the transmission dynamics of HIV/AIDS in a two-sex population considering counseling and Antiretroviral Therapy, ART. Their results reveal that the control and eradication of HIV/AIDS in heterosexual populations is dependent on the net transmission rates of the infection, the effectiveness of counseling and ART, proportion of infected people receiving ART for each sex [4,5].

In this study, we are going to use the variational iteration method to solve the systems of nonlinear ordinary differential equations in the model of Kimbir *et al.* (2012) for a two-sex HIV/AIDS modelling considering counseling and ART.

2. Methods

2.1. Systems of Ordinary Differential Equations

A number of coupled ordinary differential equations form a system of ordinary differential equations. If \mathbf{y} is a vector whose elements are functions $\mathbf{y}(t) = [y_1(t), y_2(t), \dots, y_m(t)]$, and f is a vector valued function of \mathbf{y} given as $\mathbf{f} = (f_1, f_2, \dots, f_m)$ and its derivatives, then

$$\mathbf{y}^{(n)} = \mathbf{f}(t, \mathbf{y}, \mathbf{y}', \mathbf{y}'', \dots, \mathbf{y}^{(n-1)}) \tag{2.1}$$

is an explicit system of ordinary differential equations of order n or dimension m . In column vector form:

$$\begin{pmatrix} y_1^{(n)} \\ y_2^{(n)} \\ \vdots \\ y_m^{(n)} \end{pmatrix} = \begin{pmatrix} f_1(t, y, y', y'', \dots, y^{(n-1)}) \\ f_2(t, y, y', y'', \dots, y^{(n-1)}) \\ \vdots \\ f_m(t, y, y', y'', \dots, y^{(n-1)}) \end{pmatrix}. \tag{2.2}$$

These are not necessarily linear. The implicit analogue is:

$$\mathbf{f}(t, \mathbf{y}, \mathbf{y}', \mathbf{y}'', \dots, \mathbf{y}^{(n)}) = 0 \tag{2.3}$$

Where $\mathbf{0} = (0, 0, \dots, 0)$ is the zero vector.

In matrix form

$$\begin{pmatrix} f_1(t, y, y', y'', \dots, y^{(n-1)}) \\ f_2(t, y, y', y'', \dots, y^{(n-1)}) \\ \vdots \\ f_m(t, y, y', y'', \dots, y^{(n-1)}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.4}$$

We are interested in the system of first order ordinary differential equations which can be written as

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_m' \end{pmatrix} = \begin{pmatrix} f_1(t, y, y_1, y_2, \dots, y_m) \\ f_2(t, y, y_1, y_2, \dots, y_m) \\ \vdots \\ f_m(t, y, y_1, y_2, \dots, y_m) \end{pmatrix} \tag{2.5}$$

In compact form, (2.5) can be written as

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}). \tag{2.6}$$

If in (2.6), $\mathbf{f} = \mathbf{f}(\mathbf{y})$ is explicitly independent of t , then (2.6) is said to be autonomous; otherwise it is said to be non-autonomous.

2.2. A Two-sex Model for HIV/AIDS Considering Counseling and Antiretroviral Therapy (ART)

The two-sex mathematical model for HIV/AIDS considering counseling and ART was formulated by Kimbir *et al.* (2012). This model extends the one-sex mathematical model for HIV/AIDS considering counseling and ART by Kimbir and Oduwole (2008).

2.2.1. Parameters, Variables and Flow Diagram of the Model

The parameters and variables of the model are as follows:

Table 1. Model Parameters and Variables

Variable	Parameter / Description
$S_m(t)$	Number of susceptible males at time t.
$I_m(t)$	Number of infected males at time t.
$R_m(t)$	Number of infected males receiving ART at time t.
$c_m(t)$	Average number of sexual partners of infected males with females per unit time.
$c_m^*(t)$	Average number of sexual partners of infected males receiving ART with females per unit time.
β_m	Probability of transmission by an infected male.
β_m^*	Probability of transmission by an infected male receiving ART.
σ_m	Proportion of infected males receiving ART per unit time.
$B_m(t)$	The infection rate of males at time t (incidence rate of males).
T	Maximum life-span of both sexes after infection.
K	Efficacy of ART for both sexes per unit time.
μ	Population death rate, $\mu > 0$.
α	Population death rate of infected receiving ART
α_0	Population death rate of infected not receiving ART
$S_f(t)$	Number of susceptible females at time t.
$I_f(t)$	Number of infected females at time t.
$R_f(t)$	Number of infected females receiving ART at time t.
$c_f(t)$	Average number of sexual partners of infected females with males per unit time.
$c_f^*(t)$	Average number of sexual partners of infected females receiving ART with males per unit time.
β_f	Probability of transmission by an infected female.
β_f^*	Probability of transmission by an infected female receiving ART.
σ_f	Proportion of infected females receiving ART per unit time.
$B_f(t)$	The infection rate of females at time t (incidence rate of females).

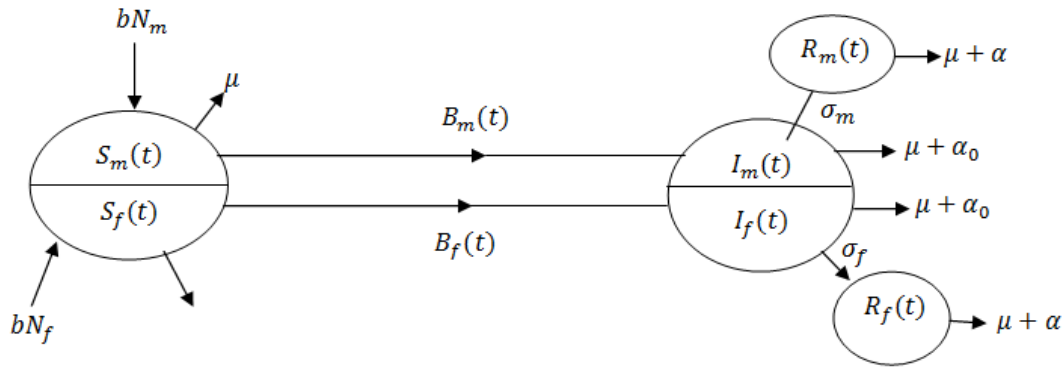


Figure 1. The flow diagram for the model of Kimbir et al. (2012)

2.2.2. Assumptions of the Model

According to Kimbir *et al.* (2012), the following assumptions are made:

1. The population is homogeneous;
2. Vertical transmission and age-structure are ignored;
3. Infected persons in addition to dying naturally;
4. The control measure is of varying efficiency;
5. The population is heterosexual (that is males and females);
6. The control measure may not be totally efficacious since infected individuals that ate on ART may transmit the infections;
7. Transmission is by heterosexual contact only;
8. Natural birth and death rates are the same for all sexes;
9. Both infected males and females use the control measure – ART;
10. Age structure is ignored.

In addition, Table 1 above gives all the parameters and variables used in the model. The flow diagram in Figure 2 will be found useful in the derivation of the model equations.

2.3. Model Equations

From the assumptions stated above and the flow diagram in Figure 1, the following models equations are derived.

$$\frac{dS_m}{dt} = bN_m - B_m S_m - \mu S_m \tag{2.7}$$

$$\frac{dI_m}{dt} = B_m S_m - (\mu + \alpha_0 + \sigma_m) I_m \tag{2.8}$$

$$\frac{dR_m}{dt} = \sigma_m I_m - (\mu + \alpha) R_m \tag{2.9}$$

$$\frac{dS_f}{dt} = bN_f - B_f S_f - \mu S_f \tag{2.10}$$

$$\frac{dI_f}{dt} = B_f S_f - (\mu + \alpha_0 + \sigma_f) I_f \tag{2.11}$$

$$\frac{dR_f}{dt} = \sigma_f I_f - (\mu + \alpha) R_f. \tag{2.12}$$

Moreover, the total population of males and females N_m and N_f are given by

$$N_m = S_m + I_m + R_m \tag{2.13}$$

$$N_f = S_f + I_f + R_f \tag{2.14}$$

Following Hseih (2003), the incidence rates for the male and female populations denoted $B_m(t)$ and $B_f(t)$ are given as

$$B_m(t) = \frac{c_m \beta_f I_f + c_m^* \beta_f^* R_f}{N_f}. \tag{2.15}$$

$$B_f(t) = \frac{c_f \beta_m I_m + c_f^* \beta_m^* R_m}{N_m}. \tag{2.16}$$

Finally, we note that $\alpha = \alpha_0 e^{-KT}$.

Transforming the model equations into proportions and the proportions of infected individuals define prevalence of infection, which has biological meaning. To achieve this, we let

$$s_m = \frac{S_m}{N_m}, i_m = \frac{I_m}{N_m} \text{ and } r_m = \frac{R_m}{N_m}.$$

So that

$$s_m + i_m + r_m = 1.$$

Likewise,

$$s_f = \frac{S_f}{N_f}, i_f = \frac{I_f}{N_f} \text{ and } r_f = \frac{R_f}{N_f}.$$

So that

$$s_f + i_f + r_f = 1.$$

To this end, the reduced system of governing equations is given as

$$\begin{aligned} \frac{di_m}{dt} = & (c_m \beta_f i_f + c_m^* \beta_f^* r_f)(1 - i_m - r_m) \\ & - (b_0 + \sigma_m) i_m + \alpha i_m r_m + \alpha i_m^2. \end{aligned} \tag{2.17}$$

$$\begin{aligned} \frac{di_f}{dt} = & (c_f \beta_m i_m + c_f^* \beta_m^* r_m)(1 - i_f - r_f) \\ & - (b_0 + \sigma_f) i_f + \alpha i_f r_f + \alpha i_f^2. \end{aligned} \tag{2.18}$$

$$\frac{dr_m}{dt} = \sigma_m i_m - (b + \alpha) r_m + \alpha_0 i_m r_m + \alpha r_m^2. \tag{2.19}$$

$$\frac{dr_f}{dt} = \sigma_f i_f - (b + \alpha) r_f + \alpha_0 i_f r_f + \alpha r_f^2 \tag{2.20}$$

Where $b_0 = b + \alpha_0$

2.3.1. The Sub-models of the General Model

Here, we shall obtain the sub-models of the general model (3.17)-(3.20) under three cases:

Case I: The sub-model without ART ($\sigma_m = \sigma_f = 0$).

In this case, $c_m^* = c_f^* = \beta_m^* = \beta_f^* = r_m = r_f = 0$, and the general model (3.17)-(3.20) reduces to

$$\frac{di_m}{dt} = c_m \beta_f i_f (1 - i_m) - b_0 i_m + \alpha_0 i_m^2, \quad (2.21)$$

$$\frac{di_f}{dt} = c_f \beta_m i_m (1 - i_f) - b_0 i_f + \alpha_0 i_f^2. \quad (2.22)$$

Case II: The sub-model with only infected males receiving ART ($\sigma_m > 0, \sigma_f = 0$).

In this case, $c_f^* = \beta_f^* = r_f = 0$, and the general model (3.17)-(3.20) reduces to

$$\begin{aligned} \frac{di_m}{dt} &= c_m \beta_f i_f (1 - i_m - r_m) \\ &\quad - (b_0 + \sigma_m) i_m + \alpha_0 i_m r_m + \alpha_0 i_m^2, \end{aligned} \quad (2.23)$$

$$\frac{di_f}{dt} = c_f \beta_m i_m (1 - i_f) - b_0 i_f + \alpha_0 i_f^2, \quad (2.24)$$

$$\frac{dr_m}{dt} = \sigma_m i_m - (b + \alpha) r_m + \alpha_0 i_m r_m + \alpha r_m^2. \quad (2.25)$$

Case III: The sub-model with only infected females receiving ART ($\sigma_m = 0, \sigma_f > 0$).

In this case, $c_m^* = \beta_m^* = r_m = 0$, and the general model (3.17)-(3.20) reduces to

$$\frac{di_m}{dt} = c_m \beta_f i_f (1 - i_m) - b_0 i_m + \alpha_0 i_m^2, \quad (2.26)$$

$$\begin{aligned} \frac{di_f}{dt} &= c_f \beta_m i_m (1 - i_f - r_f) - (b_0 + \sigma_f) i_f \\ &\quad + \alpha i_f r_f + \alpha_0 i_f^2, \end{aligned} \quad (2.27)$$

$$\frac{dr_f}{dt} = \sigma_f i_f - (b + \alpha) r_f + \alpha_0 i_f r_f + \alpha r_f^2. \quad (2.28)$$

2.3.2. He's Variational Iteration Method

To describe the Variational Iteration Method (VIM) developed by He (1999), we consider a differential equation of the form:

$$Lu(t) + N\tilde{u}(t) = g(t) \quad (2.29)$$

where L and N are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous or forcing term.

According to the variational iteration method, we can construct a correction functional as follow:

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^t \lambda \{ Lu_n(s) + N\tilde{u}_n(s) - g(s) \} ds \quad (2.30)$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, the second term on the right is called the correction, the subscript n denotes the n th approximation and \tilde{u}_n is considered as a restricted variation which means $\delta \tilde{u}_n(t) = 0$. It is required first to determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The series approximation $u_{n+1}(t)$, $n \geq 0$ of the solution $u(t)$ will be obtained upon using the Lagrange multiplier obtained and by using any initial value u_0 . The initial approximation can be freely chosen with possible unknowns, which can be determined by imposing the boundary/initial conditions.

Having λ determined, then several approximations $u_j(t)$, $j \geq 0$, can be determined. Consequently, the solution is given by

$$u(t) = \lim_{n \rightarrow \infty} u_n(t). \quad (2.31)$$

In fact, the solution of the differential equation (3.29) is considered as the fixed point of the functional (3.30) under suitable choice of the initial approximation. For the convergence proof of (3.30), we state the following known result that is useful to support the convergence of our iteration.

Theorem 1: (Bryan and Martin, 2008)

For a Banach space X , suppose the nonlinear mapping $A: X \rightarrow X$ satisfy

$$\|A[u] - A[\tilde{u}]\| \leq \gamma \|u - \tilde{u}\|, u, \tilde{u} \in X.$$

For some constants $\gamma < 1$. Then A has a unique fixed point. Furthermore, the sequence $u_{n+1} = A[u_n]$ with arbitrary choice of $u_0 \in X$, converges to the fixed point of A , and

$$u_k - u_j \leq \gamma u_1 - u_0 \sum_{i=j-1}^{k-2} \gamma^i.$$

According to this theorem, for the nonlinear mapping

$$A[u] = u(t) + \int_0^t [Lu(s) + Nu(s) - g(s)] ds.$$

A sufficient condition for the convergence of the variational iteration method is strictly contraction of A . Furthermore, the sequence (3.30) converges to the fixed point of A , which is also the solution of the differential equation in (3.29). In what follows, we will apply the variational iteration method to solve the HIV model of Kimbir *et al.* (2012), to illustrate the strength of the method and to establish approximations of high accuracy for this model.

2.3.3. Variational Iteration Method for Systems of Ordinary Differential Equations

According to He (1999), the variational iteration method applied to a system of ordinary differential equations:

$$\begin{aligned} L_1(u_1, u_2, \dots, u_m) + N_1(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m) &= g_1, \\ L_2(u_1, u_2, \dots, u_m) + N_2(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m) &= g_2, \\ &\vdots \\ L_m(u_1, u_2, \dots, u_m) + N_m(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_m) &= g_m. \end{aligned} \quad (2.32)$$

where L_1, L_2, \dots, L_m are linear operators, N_1, N_2, \dots, N_m are nonlinear operators, $u_m = u_m(t, y)$ and $g_m = g_m(t, y)$.

We can construct a correction functional for (3.32) as follows:

$$\begin{aligned} u_{1,n+1} &= u_{1,n} + \int_{t_0}^t \lambda_1 \left[L_1(u_{1,n}, u_{2,n}, \dots, u_{m,n}) \right. \\ &\quad \left. + N_1(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \dots, \tilde{u}_{m,n}) - g_1 \right] ds, \\ u_{2,n+1} &= u_{2,n} + \int_{t_0}^t \lambda_2 \left[L_2(u_{1,n}, u_{2,n}, \dots, u_{m,n}) \right. \\ &\quad \left. + N_2(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \dots, \tilde{u}_{m,n}) - g_2 \right] ds, \\ &\vdots \\ u_{m,n+1} &= u_{m,n} + \int_{t_0}^t \lambda_m \left[L_m(u_{1,n}, u_{2,n}, \dots, u_{m,n}) \right. \\ &\quad \left. + N_m(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \dots, \tilde{u}_{m,n}) - g_m \right] ds. \end{aligned} \quad (2.33)$$

where $\lambda_1, \lambda_2, \dots, \lambda_m$ are a general Lagrange multipliers which can be identified optimally via the variational theory, the subscript n denotes the n th – order approximation, \tilde{u}_n is considered as restricted variation i.e. $\delta \tilde{u}_n = 0$.

By omitting the restrictions, (3.33) becomes

$$\begin{aligned} u_{1,n+1}(t) &= u_{1,n}(t) + \int_0^t \lambda_1 L_1 \{u_{1,n}(s)\} ds \\ u_{2,n+1}(t) &= u_{2,n}(t) + \int_0^t \lambda_2 L_2 \{u_{2,n}(s)\} ds \\ &\vdots \\ u_{m,n+1}(t) &= u_{m,n}(t) + \int_0^t \lambda_m L_m \{u_{m,n}(s)\} ds \end{aligned} \tag{2.34}$$

Assuming that all the linear operators are of first order ordinary differential equations.

Now, by integrating (3.34) yields

$$\begin{aligned} u_{1,n+1}(t) &= u_{1,n}(t) + \lambda_1 u_{1,n}(t) - \int_0^t \lambda_1' u_{1,n}(s) ds \\ u_{2,n+1}(t) &= u_{2,n}(t) + \lambda_2 u_{2,n}(t) - \int_0^t \lambda_2' u_{2,n}(s) ds \\ &\vdots \\ u_{m,n+1}(t) &= u_{m,n}(t) + \lambda_m u_{m,n}(t) - \int_0^t \lambda_m' u_{m,n}(s) ds \end{aligned} \tag{2.35}$$

Taking the variational derivative on both sides of (3.35) gives

$$\begin{aligned} \delta u_{1,n+1}(t) &= \delta u_{1,n}(s)(1 + \lambda_1 |_{s=t}) \\ &\quad - \int_0^t \lambda_1' \delta u_{1,n}(s) ds = 0 \\ \delta u_{2,n+1}(t) &= \delta u_{2,n}(s)(1 + \lambda_2 |_{s=t}) \\ &\quad - \int_0^t \lambda_2' \delta u_{2,n}(s) ds = 0 \\ &\vdots \\ \delta u_{m,n+1}(t) &= \delta u_{m,n}(s)(1 + \lambda_m |_{s=t}) \\ &\quad - \int_0^t \lambda_m' \delta u_{m,n}(s) ds = 0 \end{aligned} \tag{2.36}$$

The extremum condition of each $u_{m,n+1}$ requires that $\delta u_{m,n+1} = 0$.

This yields the stationary conditions from (3.36) as

$$\begin{aligned} 1 + \lambda_1 |_{s=t} &= 0, \lambda_1' |_{s=t} = 0 \\ 1 + \lambda_2 |_{s=t} &= 0, \lambda_2' |_{s=t} = 0 \\ &\vdots \\ 1 + \lambda_m |_{s=t} &= 0, \lambda_m' |_{s=t} = 0 \end{aligned}$$

These in turn give

$$\lambda_1 = \lambda_2 = \dots = \lambda_m = -1 \tag{2.37}$$

Substituting the values of Lagrange multipliers into (3.30) gives the iteration

$$\begin{aligned} u_{1,n+1} &= u_{1,n} - \int_0^t \left[L_1(u_{1,n}, u_{2,n}, \dots, u_{m,n}) \right. \\ &\quad \left. + N_1(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \dots, \tilde{u}_{m,n}) - g_1 \right] ds, \\ u_{2,n+1} &= u_{2,n} - \int_0^t \left[L_2(u_{1,n}, u_{2,n}, \dots, u_{m,n}) \right. \\ &\quad \left. + N_2(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \dots, \tilde{u}_{m,n}) - g_2 \right] ds, \\ &\vdots \\ u_{m,n+1} &= u_{m,n} - \int_0^t \left[L_m(u_{1,n}, u_{2,n}, \dots, u_{m,n}) \right. \\ &\quad \left. + N_m(\tilde{u}_{1,n}, \tilde{u}_{2,n}, \dots, \tilde{u}_{m,n}) - g_m \right] ds. \end{aligned} \tag{2.38}$$

At this point, the series approximations $u_{1,n+1}, u_{2,n+1}, \dots, u_{m,n+1}$ for $m > 0, n \geq 0$ of the solution $u(t)$ will be obtained using the initial values $u_{1,0}, u_{2,0}, \dots, u_{m,0}$.

3. Results and Discussion

3.1. Model Analysis

In this research, the general model (2.17)-(2.20) is analyzed to investigate the existence of equilibrium state and its stability.

3.1.1. Disease Free Equilibrium State

At equilibrium,

$$\frac{di_m}{dt} = \frac{di_f}{dt} = \frac{dr_m}{dt} = \frac{dr_f}{dt} = 0$$

Which implies

$$\begin{aligned} 0 &= (c_m \beta_f i_f + c_m^* \beta_f^* r_f)(1 - i_m - r_m) \\ &\quad - (b_0 + \sigma_m) i_m + \alpha i_m r_m + \alpha i_m^2, \\ 0 &= (c_f \beta_m i_m + c_f^* \beta_m^* r_m)(1 - i_f - r_f) \\ &\quad - (b_0 + \sigma_f) i_f + \alpha i_f r_f + \alpha i_f^2 \\ 0 &= \sigma_m i_m - (b + \alpha) r_m + \alpha_0 i_m r_m + \alpha r_m^2, \\ 0 &= \sigma_f i_f - (b + \alpha) r_f + \alpha_0 i_f r_f + \alpha r_f^2, \end{aligned} \tag{3.1}$$

At this state, we assume that there is no infection. That means $i_m = i_f = r_m = r_f = 0$.

The substitution of this condition in (4.1), we see that $E_0 = (i_m, i_f, r_m, r_f) = (0, 0, 0, 0)$ is the disease-free equilibrium state (DFES) of the model.

3.1.2. Stability Analysis of the Disease Free Equilibrium State of the Model

Here $c_m^* = c_f^* = \beta_m^* = \beta_f^* = 0$.

To determine the stability of the equilibrium state by the linearization stability method, let

$$\begin{aligned} f_1 &= c_m \beta_f i_f (1 - i_m - r_m) - (b_0 + \sigma_m) i_m + \alpha i_m r_m + \alpha i_m^2, \\ f_2 &= c_f \beta_m i_m (1 - i_f - r_f) - (b_0 + \sigma_f) i_f + \alpha i_f r_f + \alpha i_f^2 \\ f_3 &= \sigma_m i_m - (b + \alpha) r_m + \alpha_0 i_m r_m + \alpha r_m^2, \\ f_4 &= \sigma_f i_f - (b + \alpha) r_f + \alpha_0 i_f r_f + \alpha r_f^2, \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial f_1}{\partial i_m} &= -c_m \beta_f i_f - (b_0 + \sigma_m) + \alpha r_m + 2\alpha i_m, \\ \frac{\partial f_1}{\partial i_f} &= c_m \beta_f (1 - i_m - r_m), \\ \frac{\partial f_1}{\partial r_m} &= -c_m \beta_f i_f + \alpha i_m, \\ \frac{\partial f_1}{\partial r_f} &= 0, \end{aligned}$$

$$\begin{aligned}
 \frac{\partial f_2}{\partial i_m} &= c_f \beta_m (1 - i_f - r_f), & \frac{\partial f_3}{\partial r_m} &= -(b + \alpha) + \alpha_0 i_m + 2\alpha r_m, \\
 \frac{\partial f_2}{\partial i_f} &= -c_f \beta_m i_m - (b_0 + \sigma_f) + \alpha r_f + 2\alpha i_f, & \frac{\partial f_3}{\partial r_f} &= 0, \\
 \frac{\partial f_2}{\partial r_m} &= 0, & \frac{\partial f_4}{\partial r_f} &= 0, \\
 \frac{\partial f_2}{\partial r_f} &= -c_f \beta_m i_m + \alpha i_f, & \frac{\partial i_m}{\partial r_m} &= 0, \\
 \frac{\partial f_3}{\partial i_m} &= \sigma_m + \alpha_0 r_m, & \frac{\partial f_4}{\partial i_f} &= \sigma_f + \alpha_0 r_f, \\
 \frac{\partial f_3}{\partial i_f} &= 0, & \frac{\partial f_4}{\partial r_m} &= 0, \\
 & & \frac{\partial f_4}{\partial r_f} &= -(b + \alpha) + \alpha_0 i_f + 2\alpha r_f
 \end{aligned}$$

Therefore the Jacobian matrix is

$$J = \begin{pmatrix} \begin{bmatrix} -c_m \beta_f i_f - (b_0 + \sigma_m) \\ +\alpha r_m + 2\alpha i_m \end{bmatrix} & c_m \beta_f (1 - i_m - r_m) & -c_m \beta_f i_f + \alpha i_m & 0 \\ c_f \beta_m (1 - i_f - r_f) & \begin{bmatrix} -c_f \beta_m i_m - (b_0 + \sigma_f) \\ +\alpha r_f + 2\alpha i_f \end{bmatrix} & 0 & -c_f \beta_m i_m + \alpha i_f \\ \sigma_m + \alpha_0 r_m & 0 & \begin{bmatrix} -(b + \alpha) + \alpha_0 i_m \\ +2\alpha r_m \end{bmatrix} & 0 \\ 0 & \sigma_f + \alpha_0 r_f & 0 & \begin{bmatrix} -(b + \alpha) \\ +\alpha_0 i_f + 2\alpha r_f \end{bmatrix} \end{pmatrix} \tag{3.2}$$

Now, corresponding to the disease –free equilibrium state $E_0 = (0, 0, 0, 0)$, the Jacobian J at E_0 is given by

$$J(E_0) = \begin{pmatrix} -(b_0 + \sigma_m) & c_m \beta_f & 0 & 0 \\ c_f \beta_m & -(b_0 + \sigma_f) & 0 & 0 \\ \sigma_m & 0 & -(b + \alpha) & 0 \\ 0 & \sigma_f & 0 & -(b + \alpha) \end{pmatrix} \tag{3.3}$$

With the corresponding characteristic equation given as

$$|J(E_0) - \lambda I| = \begin{vmatrix} -(b_0 + \sigma_m) - \lambda & c_m \beta_f & 0 & 0 \\ c_f \beta_m & -(b_0 + \sigma_f) - \lambda & 0 & 0 \\ \sigma_m & 0 & -(b + \alpha) - \lambda & 0 \\ 0 & \sigma_f & 0 & -(b + \alpha) - \lambda \end{vmatrix} = 0$$

By partitioning the above characteristic equation, we have

$$\begin{vmatrix} ((b_0 + \sigma_m) + \lambda)((b_0 + \sigma_f) + \lambda) - c_m c_f \beta_m \beta_f & 0 \\ \sigma_m \sigma_f & ((b + \alpha) + \lambda)((b + \alpha) + \lambda) \end{vmatrix} = 0$$

$$[((b_0 + \sigma_m) + \lambda)((b_0 + \sigma_f) + \lambda) - c_m c_f \beta_m \beta_f][((b + \alpha) + \lambda)((b + \alpha) + \lambda)] = 0 \tag{3.4}$$

From which we obtain

$$\lambda_1 = \lambda_2 = -(b + \alpha) < 0.$$

And λ_3, λ_4 are found from the quadratic equation below

$$\lambda^2 + (2b_0 + \sigma_m + \sigma_f)\lambda + (b_0 + \sigma_m)(b_0 + \sigma_f) - c_m c_f \beta_m \beta_f = 0$$

Or

$$\lambda^2 + a\lambda + a_0 = 0 \quad (3.5)$$

Where

$$a = 2b_0 + \sigma_m + \sigma_f$$

$$a_0 = (b_0 + \sigma_m)(b_0 + \sigma_f) - c_m c_f \beta_m \beta_f$$

Applying Routh-Hurwitz criteria on (3.5), we have that $a, a_0 > 0$, provided $c_m c_f \beta_m \beta_f \leq 0$ and

$$a \cdot a_0 = [(b_0 + \sigma_m)(b_0 + \sigma_f)(2b_0 + \sigma_m + \sigma_f)] - c_m c_f \beta_m \beta_f (2b_0 + \sigma_m + \sigma_f) > 0.$$

Since Routh Hurwitz conditions are satisfied, we can conclude that the eigenvalues λ_3, λ_4 are real and negative, and hence, the HIV/AIDS free equilibrium state $E_0 = (0, 0, 0, 0)$ is locally and asymptotically stable (LAS).

3.2. Solution of the Model Equations by the Variational Iteration Method

We will solve the general model and the sub-models using the variational iteration method and then compare the solution with those obtained from the Runge-Kutta method. The Runge-Kutta method used is the inbuilt ode 45 algorithm of MATLAB. We will consider the six distinct situations.

3.2.1. Solution of the sub-model without ART

Consider (2.21)-(2.22) with parameters

$$b = 0.5, \quad \alpha_0 = 0.2, \quad c_m = 4, \quad c_f = 4,$$

$$\beta_m = 0.35, \quad \beta_f = 0.35, \quad b_0 = b + \alpha_0 = 0.7$$

and initial conditions $i_m(0) = 0.35, i_f(0) = 0.35$ which yields

$$i'_m = 1.4 i_f(1 - i_m) - 0.7 i_m + 0.2 i_m^2, \quad i_m(0) = 0.35 \quad (3.6)$$

$$i'_f = 1.4 i_m(1 - i_f) - 0.7 i_f + 0.2 i_f^2, \quad i_f(0) = 0.35$$

The correction functional for (3.6) reads

$$\begin{aligned} i_{m,n+1} &= i_{m,n} + \int_0^t \lambda_1 \{i'_{m,n}(s) - 1.4 i_{f,n}(1 - i_{m,n}) + 0.7 i_{m,n} - 0.2 i_{m,n}^2\} ds, \\ i_{f,n+1} &= i_{f,n} + \int_0^t \lambda_2 \{i'_{f,n}(s) - 1.4 i_{m,n}(1 - i_{f,n}) + 0.7 i_{f,n} - 0.2 i_{f,n}^2\} ds \end{aligned} \quad (3.7)$$

where $n \geq 0$.

By integrating yields

$$\begin{aligned} i_{m,n+1} &= (1 + \lambda_1) i_{m,n} - \int_0^t (\lambda'_1 - 0.7 \lambda_1) i_{m,n}(s) ds, \\ i_{f,n+1} &= (1 + \lambda_2) i_{f,n} - \int_0^t (\lambda'_2 - 0.7 \lambda_2) i_{f,n}(s) ds \end{aligned}$$

Taking variational derivative on both sides, we have

$$\begin{aligned} \delta i_{m,n+1} &= (1 + \lambda_1) \delta i_{m,n} - \int_0^t (\lambda'_1 - 0.7 \lambda_1) \delta i_{m,n}(s) ds = 0, \\ \delta i_{f,n+1} &= (1 + \lambda_2) \delta i_{f,n} - \int_0^t (\lambda'_2 - 0.7 \lambda_2) \delta i_{f,n}(s) ds = 0 \end{aligned}$$

This yields the stationary conditions

$$\begin{aligned} 1 + \lambda_1(s) &= 0, & \lambda'_1|_{s=t} - 0.7 \lambda_1(s) &= 0 \\ 1 + \lambda_2(s) &= 0, & \lambda'_2|_{s=t} - 0.7 \lambda_2(s) &= 0, \end{aligned}$$

As a result, we find

$$\lambda_1(s) = \lambda_2(s) = -e^{0.7(s-t)} \quad (3.8)$$

Substituting (3.8) into (3.7) gives

$$\begin{aligned} i_{m,n+1} &= i_{m,n} - \int_0^t e^{0.7(s-t)} \{i'_{m,n}(s) - 1.4 i_{f,n}(1 - i_{m,n}) + 0.7 i_{m,n} - 0.2 i_{m,n}^2\} ds, \\ i_{f,n+1} &= i_{f,n} - \int_0^t e^{0.7(s-t)} \{i'_{f,n}(s) - 1.4 i_{m,n}(1 - i_{f,n}) + 0.7 i_{f,n} - 0.2 i_{f,n}^2\} ds \end{aligned} \tag{3.9}$$

Accordingly, we obtain the following series approximations using maple.

$$\begin{aligned} i_{m0} &= 0.35, & i_{f0} &= 0.35 \\ i_{m1} &= 0.49 - 0.14e^{-0.7t}, & i_{f1} &= 0.49 - 0.14e^{-0.7t} \\ i_{m2} &= 0.5684 - 0.1232e^{-0.7t} - 0.1288e^{-1.4t} + 0.0336e^{-2.1t} \\ i_{f2} &= 0.5684 - 0.1232e^{-0.7t} - 0.1288e^{-1.4t} + 0.0336e^{-2.1t} \\ & \vdots \end{aligned}$$

3.2.2. Solution of the Sub-model with Only Infected Males Receiving ART

Consider (2.23)-(2.25) with parameters

$$b = 0.5, \quad \alpha_0 = 0.2, \quad \sigma_m = 0.3, \quad \sigma_f = 0.3, \quad c_m = 3, \quad c_f = 3,$$

$$\beta_m = 0.35, \quad \beta_f = 0.35, \quad b_0 = b + \alpha_0 = 0.7$$

$$T = 10, \quad k = 0.1, \quad b_0 + \sigma_m = 1, \quad \alpha = \alpha_0 e^{-kT} = 0.07$$

and initial conditions $i_m(0) = 0.35, \quad i_f(0) = 0.35, \quad r_m(0) = 0$ which yields

$$i'_m = 1.05i_f(1 - i_m - r_m) - i_m + 0.7i_m r_m + 0.2i_m^2, \quad i_m(0) = 0.35$$

$$i'_f = 1.05i_m(1 - i_f) - 0.7i_f + 0.2i_f^2, \quad i_f(0) = 0.35 \tag{3.11}$$

$$r'_m = 0.3i_m - 0.57r_m + 0.2i_m r_m + 0.07r_m^2, \quad r_m(0) = 0$$

The correction functional for (3.11) reads

$$\begin{aligned} i_{m,n+1} &= i_{m,n} + \int_0^t \lambda_1 \{i'_{m,n} - 1.05i_{f,n}(1 - i_{m,n} - r_{m,n}) + i_{m,n} - 0.7i_{m,n}r_{m,n} - 0.2i_{m,n}^2\} ds, \\ i_{f,n+1} &= i_{f,n} + \int_0^t \lambda_2 \{i'_{f,n} - 1.05i_{m,n}(1 - i_{f,n}) + 0.7i_{f,n} + 0.2i_{f,n}^2\} ds, \\ r_{m,n+1} &= r_{m,n} + \int_0^t \lambda_3 \{r'_{m,n} - 0.3i_{m,n} + 0.57r_{m,n} - 0.2i_{m,n}r_{m,n} - 0.07r_{m,n}^2\} ds \end{aligned} \tag{3.12}$$

Where $n \geq 0$.

Integrating (3.12) yields

$$\begin{aligned} i_{m,n+1} &= (1 + \lambda_1)i_{m,n}(s) - \int_0^t (\lambda'_1 - \lambda_1)i_{m,n}(s) ds, \\ i_{f,n+1} &= (1 + \lambda_2)i_{f,n}(s) - \int_0^t (\lambda'_2 - 0.7\lambda_2)i_{f,n}(s) ds, \\ r_{m,n+1} &= (1 + \lambda_3)r_{m,n}(s) - \int_0^t (\lambda'_3 - 0.57\lambda_3)r_{m,n}(s) ds \end{aligned}$$

Taking the variational derivative on both sides yields

$$\begin{aligned} \delta i_{m,n+1} &= \delta(1 + \lambda_1)i_{m,n}(s) - \delta \int_0^t (\lambda'_1 - \lambda_1)i_{m,n}(s) ds = 0, \\ \delta i_{f,n+1} &= \delta(1 + \lambda_2)i_{f,n}(s) - \delta \int_0^t (\lambda'_2 - 0.7\lambda_2)i_{f,n}(s) ds = 0, \\ \delta r_{m,n+1} &= \delta(1 + \lambda_3)r_{m,n}(s) - \delta \int_0^t (\lambda'_3 - 0.57\lambda_3)r_{m,n}(s) ds = 0 \end{aligned}$$

This yields the stationary conditions

$$\begin{aligned} 1 + \lambda_1(s) &= 0, & \lambda'_1|_{s=t} - \lambda_1(s) &= 0, \\ 1 + \lambda_2(s) &= 0, & \lambda'_2|_{s=t} - 0.7\lambda_2(s) &= 0, \\ 1 + \lambda_3(s) &= 0 & \lambda'_3|_{s=t} - 0.57\lambda_3(s) &= 0 \end{aligned}$$

As a result, we find

$$\lambda_1(s) = -e^{s-t}, \lambda_2(s) = -e^{0.7(s-t)}, \lambda_3(s) = -e^{0.57(s-t)} \tag{3.13}$$

Substituting (3.13) into (3.12) gives

$$\begin{aligned} i_{m,n+1} &= i_{m,n} - \int_0^t e^{s-t} \{i'_{m,n} - 1.05i_{f,n}(1 - i_{m,n} - r_{m,n}) + i_{m,n} - 0.7i_{m,n}r_{m,n} - 0.2i_{m,n}^2\} ds, \\ i_{f,n+1} &= i_{f,n} - \int_0^t e^{0.7(s-t)} \{i'_{f,n} - 1.05i_{m,n}(1 - i_{f,n}) + 0.7i_{f,n} - 0.2i_{f,n}^2\} ds, \\ r_{m,n+1} &= r_{m,n} - \int_0^t e^{0.57(s-t)} \{r'_{m,n} - 0.3i_{m,n} + 0.57r_{m,n} - 0.2i_{m,n}r_{m,n} - 0.07r_{m,n}^2\} ds \end{aligned} \tag{3.14}$$

Accordingly, we obtain the following series approximations:

$$\begin{aligned} i_{m0} &= 0.35, \quad i_{f0} = 0.35, \quad r_{m0} = 0 \\ i_{m1} &= 0.263375 + 0.086625e^{-t}, \\ i_{f1} &= 0.37625 - 0.02625e^{-0.7t} \\ r_{m1} &= 0.1842105263 - 0.1842105263e^{-0.57t} \\ i_{m2} &= 0.2355076725 + 0.00388798879881e^{-t} + 0.1121051169e^{-2t} - 0.07049552631e^{-1.57t} \\ &\quad + 0.01761352549e^{-1.7t} - 0.002387601562e^{-2.7t} + 0.005077302631e^{-2.27t} \\ &\quad + 0.00117006579e^{-2.57t} - 0.001500778125e^{-3t} + 0.06937851973e^{-0.57t} \\ &\quad - 0.01522592393e^{-0.7t} - 0.005077302631e^{-1.27t} \\ i_{f2} &= 0.2868671094 + 0.0941095312e^{-0.7t} - 0.03077976562e^{-1.4t} - 0.07763765625e^{-1.7t} \\ &\quad - 0.003410859375e^{-2.4t} - 0.000196875e^{-2.1t} + 0.08104851562e^{-t} \\ r_{m2} &= 0.15980901 - 0.0009563469e^{-0.57t} - 0.0567901662e^{-1.57t} - 0.154685389e^{-1.14t} \\ &\quad + 0.00559903047e^{-2.14t} - 0.00416727414e^{-1.71t} + 0.05119113573e^{-t} \\ &\quad \vdots \end{aligned}$$

3.2.3. Solution of the Sub-model with only Infected Females Receiving ART

Consider (2.26)-(2.28) with parameters

$$\begin{aligned} b &= 0.5, \quad \alpha_0 = 0.2, \quad \sigma_f = 0.3, \quad \sigma_m = 0.3, \quad c_m = 3, \quad c_f = 3, \\ \beta_f &= 0.35, \quad \beta_m = 0.35, \quad b_0 = b + \alpha_0 = 0.7, \quad b_0 + \sigma_f = 1, \quad b + \alpha = 0.57 \\ T &= 10, \quad k = 0.1, \quad \alpha = \alpha_0 e^{-kT} = 0.07 \end{aligned}$$

and initial conditions $i_m(0) = 0.35, i_f(0) = 0.35, r_f(0) = 0$ which yields

$$\begin{aligned} i'_m &= 1.05i_f(1 - i_m) - 0.7i_m + 0.2i_m^2, \quad i_m(0) = 0.35 \\ i'_f &= 1.05i_m(1 - i_f - r_f) - i_f + 0.07i_f r_f + 0.2i_f^2, \quad i_f(0) = 0.35 \\ r'_f &= 0.3i_f - 0.57r_f + 0.2i_f r_f + 0.07r_f^2, \quad r_f(0) = 0 \end{aligned} \tag{3.15}$$

The correction functional for (3.15) yields

$$\begin{aligned} i_{m,n+1} &= i_{m,n} + \int_0^t \lambda_1 \{i'_{m,n} - 1.05i_{f,n}(1 - i_{m,n}) + 0.7i_{m,n} - 0.2i_{m,n}^2\} ds, \\ i_{f,n+1} &= i_{f,n} + \int_0^t \lambda_2 \{i'_{f,n} - 1.05i_{m,n}(1 - i_{f,n} - r_{f,n}) + i_{f,n} - 0.07i_f r_f - 0.2i_{f,n}^2\} ds, \\ r_{f,n+1} &= r_{f,n} + \int_0^t \lambda_3 \{r'_{f,n} - 0.3i_{f,n} + 0.57r_{f,n} - 0.2i_{f,n}r_{f,n} - 0.07r_{f,n}^2\} ds \end{aligned} \tag{3.16}$$

where $n \geq 0$.

Integrating (3.16) yields

$$\begin{aligned} i_{m,n+1} &= (1 + \lambda_1)i_{m,n}(s) - \int_0^t (\lambda'_1 - 0.7\lambda_1)i_{m,n}(s) ds, \\ i_{f,n+1} &= (1 + \lambda_2)i_{f,n}(s) - \int_0^t (\lambda'_2 - \lambda_2)i_{f,n}(s) ds, \\ r_{f,n+1} &= (1 + \lambda_3)r_{f,n}(s) - \int_0^t (\lambda'_3 - 0.57\lambda_3)r_{f,n}(s) ds \end{aligned}$$

Taking the variational derivative on both sides yields

$$\begin{aligned} \delta i_{m,n+1} &= \delta(1 + \lambda_1)i_{m,n}(s) - \delta \int_0^t (\lambda'_1 - 0.7\lambda_1)i_{m,n}(s) ds = 0, \\ \delta i_{f,n+1} &= \delta(1 + \lambda_2)i_{f,n}(s) - \delta \int_0^t (\lambda'_2 - \lambda_2)i_{f,n}(s) ds = 0, \\ \delta r_{f,n+1} &= \delta(1 + \lambda_3)r_{f,n}(s) - \delta \int_0^t (\lambda'_3 - 0.57\lambda_3)r_{f,n}(s) ds = 0 \end{aligned}$$

This yields the stationary conditions

$$\begin{aligned} 1 + \lambda_1(s) &= 0, & \lambda'_1|_{s=t} - 0.7 \lambda_1(s) &= 0, \\ 1 + \lambda_2(s) &= 0, & \lambda'_2|_{s=t} - \lambda_2(s) &= 0, \\ 1 + \lambda_3(s) &= 0 & \lambda'_3|_{s=t} - 0.57 \lambda_3(s) &= 0 \end{aligned}$$

As a result, we find

$$\lambda_1(s) = -e^{0.7(s-t)}, \lambda_2(s) = -e^{(s-t)}, \lambda_3(s) = -e^{0.57(s-t)} \tag{3.17}$$

Substituting (3.17) into (3.16) gives

$$\begin{aligned} i_{m,n+1} &= i_{m,n} - \int_0^t e^{0.7(s-t)} \{i'_{m,n} - 1.05i_{f,n}(1 - i_{m,n}) + 0.7i_{m,n} - 0.2i_{m,n}^2\} ds, \\ i_{f,n+1} &= i_{f,n} - \int_0^t e^{(s-t)} \{i'_{f,n} - 1.05i_{m,n}(1 - i_{f,n} - r_{f,n}) + i_{f,n} - 0.07i_{f,n}r_{f,n} - 0.2i_{f,n}^2\} ds, \\ r_{f,n+1} &= r_{f,n} - \int_0^t e^{0.57(s-t)} \{r'_{f,n} - 0.3i_{f,n} + 0.57r_{f,n} - 0.2i_{f,n}r_{f,n} - 0.07r_{f,n}^2\} ds \end{aligned} \tag{3.18}$$

Accordingly, we obtain the following series approximations:

$$\begin{aligned} i_{m0} &= 0.35, & i_{f0} &= 0.35, & r_{f0} &= 0 \\ i_{m1} &= 0.37625 - 0.02625e^{-0.7t}, \\ i_{f1} &= 0.263375 + 0.086625e^{-t} \\ r_{f1} &= 0.1842105263 - 0.1842105263e^{-0.57t} \\ i_{m2} &= 0.2868671094 + 0.0941095312e^{-0.7t} - 0.02077976562e^{-1.4t} - 0.07763765625e^{-1.7t} \\ &\quad - 0.003410859375e^{-2.4t} - 0.000196875e^{-2.1t} + 0.08104851562e^{-t} \\ i_{f2} &= 0.2355076725 + 0.00388798881e^{-t} + 0.1121059169e^{-2t} - 0.07049552631e^{-1.57t} + 0.01761352549e^{-1.7t} \\ &\quad - 0.002387601562e^{-2.7t} + 0.005077302631e^{-2.27t} + 0.001117006571e^{-2.57t} \\ &\quad - 0.001500778125e^{-3t} + 0.06937851973e^{-0.57t} - 0.01522512393e^{-0.7t} \\ &\quad - 0.005077302631e^{-1.27t} \\ r_{f2} &= 0.15980901 - 0.0009563469e^{-0.57t} - 0.0567901662e^{-1.57t} - 0.154685389e^{-1.14t} + 0.00559903047e^{-2.14t} \\ &\quad - 0.00416727414e^{-1.71t} + 0.05119113573e^{-t} \\ &\quad \vdots \end{aligned}$$

3.2.4. Solution of the General Model with the Absence of Counseling and ART

Consider (2.17)-(2.20) with parameters

$$b = 0.5, \quad \alpha_0 = 0.2, \quad \sigma_f = 0, \quad \sigma_m = 0, \quad c_m = 4, \quad c_f = 4, \quad c_m^* = 0, \quad c_f^* = 0$$

$$b_0 = b + \alpha_0 = 0.7, \quad \beta_f = 0.35, \quad \beta_m = 0.35, \quad \beta_f^* = 0, \quad \beta_m^* = 0$$

$$T = 10, \quad k = 0, \quad \alpha = \alpha_0 e^{-kT} = 0.2$$

and initial conditions $i_m(0) = 0.35, i_f(0) = 0.35, r_m(0) = 0, r_f(0) = 0$ which yields

$$\begin{aligned} i'_m &= 1.4i_f(1 - i_m) - 0.7i_m + 0.2i_m^2, & i_m(0) &= 0.35 \\ i'_f &= 1.4i_m(1 - i_f - r_f) - 0.7i_f + 0.2i_f r_f + 0.2i_f^2, & i_f(0) &= 0.35 \\ r'_m &= -0.7i_m + 0.2i_m r_m + 0.2r_m^2, & r_f(0) &= 0 \\ r'_f &= -0.7i_f + 0.2i_f r_f + 0.2r_f^2, & r_f(0) &= 0 \end{aligned} \tag{3.19}$$

The correction functional for (3.19) reads

$$\begin{aligned}
 i_{m,n+1} &= i_{m,n} + \int_0^t \lambda_1 \{i'_{m,n} - 1.4i_{f,n}(1 - i_{m,n} - r_{m,n}) + 0.7i_{m,n} - 0.2i_{m,n}r_{m,n} - 0.2i_{m,n}^2\} ds, \\
 i_{f,n+1} &= i_{f,n} + \int_0^t \lambda_2 \{i'_{f,n} - 1.4i_{m,n}(1 - i_{f,n} - r_{f,n}) + 0.7i_{f,n} - 0.2i_{f,n}r_{f,n} - 0.2i_{f,n}^2\} ds, \\
 r_{m,n+1} &= r_{m,n} + \int_0^t \lambda_3 \{r'_{m,n} + 0.7r_{m,n} - 0.2i_{m,n}r_{m,n} - 0.2r_{m,n}^2\} ds \\
 r_{f,n+1} &= r_{f,n} + \int_0^t \lambda_4 \{r'_{f,n} + 0.7r_{f,n} - 0.2i_{f,n}r_{f,n} - 0.2r_{f,n}^2\} ds
 \end{aligned} \tag{3.20}$$

Integration gives

$$\begin{aligned}
 i_{m,n+1} &= (1 + \lambda_1)i_{m,n} + \int_0^t (\lambda'_1 - 0.7\lambda_1)i_{m,n}(s) ds, \\
 i_{f,n+1} &= (1 + \lambda_2)i_{f,n} + \int_0^t (\lambda'_2 - 0.7\lambda_2)i_{f,n}(s) ds, \\
 r_{m,n+1} &= (1 + \lambda_3)r_{m,n} + \int_0^t (\lambda'_3 - 0.7\lambda_3)r_{m,n}(s) ds \\
 r_{f,n+1} &= (1 + \lambda_4)r_{f,n} + \int_0^t (\lambda'_4 - 0.7\lambda_4)r_{f,n}(s) ds
 \end{aligned} \tag{3.21}$$

Taking variational derivative on both sides gives

$$\begin{aligned}
 \delta i_{m,n+1} &= (1 + \lambda_1)\delta i_{m,n} + \int_0^t (\lambda'_1 - 0.7\lambda_1)\delta i_{m,n}(s) ds = 0, \\
 \delta i_{f,n+1} &= (1 + \lambda_2)\delta i_{f,n} + \int_0^t (\lambda'_2 - 0.7\lambda_2)\delta i_{f,n}(s) ds = 0, \\
 \delta r_{m,n+1} &= (1 + \lambda_3)\delta r_{m,n} + \int_0^t (\lambda'_3 - 0.7\lambda_3)\delta r_{m,n}(s) ds = 0 \\
 \delta r_{f,n+1} &= (1 + \lambda_4)\delta r_{f,n} + \int_0^t (\lambda'_4 - 0.7\lambda_4)\delta r_{f,n}(s) ds = 0
 \end{aligned}$$

This yields the stationary conditions

$$\begin{aligned}
 1 + \lambda_1(s) &= 0, & \lambda'_1|_{s=t} - 0.7\lambda_1(s) &= 0, \\
 1 + \lambda_2(s) &= 0, & \lambda'_2|_{s=t} - 0.7\lambda_2(s) &= 0, \\
 1 + \lambda_3(s) &= 0, & \lambda'_3|_{s=t} - 0.7\lambda_3(s) &= 0 \\
 1 + \lambda_4(s) &= 0, & \lambda'_4|_{s=t} - 0.7\lambda_4(s) &= 0
 \end{aligned}$$

As a result, we find

$$\lambda_1(s) = \lambda_2(s) = \lambda_3(s) = \lambda_4(s) = -e^{0.7(s-t)} \tag{3.22}$$

Substituting (3.22) into (3.20) gives

$$\begin{aligned}
 i_{m,n+1} &= i_{m,n} - \int_0^t e^{0.7(s-t)} \{i'_{m,n} - 1.4i_{f,n}(1 - i_{m,n} - r_{m,n}) + 0.7i_{m,n} - 0.2i_{m,n}r_{m,n} - 0.2i_{m,n}^2\} ds, \\
 i_{f,n+1} &= i_{f,n} - \int_0^t e^{0.7(s-t)} \{i'_{f,n} - 1.4i_{m,n}(1 - i_{f,n} - r_{f,n}) + 0.7i_{f,n} - 0.2i_{f,n}r_{f,n} - 0.2i_{f,n}^2\} ds, \\
 r_{m,n+1} &= r_{m,n} - \int_0^t e^{0.7(s-t)} \{r'_{m,n} + 0.7r_{m,n} - 0.2i_{m,n}r_{m,n} - 0.2r_{m,n}^2\} ds \\
 r_{f,n+1} &= r_{f,n} - \int_0^t e^{0.7(s-t)} \{r'_{f,n} + 0.7r_{f,n} - 0.2i_{f,n}r_{f,n} - 0.2r_{f,n}^2\} ds
 \end{aligned}$$

Accordingly, we obtain the following series approximations:

$$\begin{aligned}
 i_{m0} &= 0.35, & i_{f0} &= 0.35, & r_{m0} &= 0, & r_{f0} &= 0 \\
 i_{m1} &= 0.49 - 0.14e^{-0.7t}, & i_{f1} &= 0.49 - 0.14e^{-0.7t}, & r_{m1} &= 0, & r_{f1} &= 0 \\
 i_{m2} &= 0.5684 - 0.1232e^{-0.7t} - 0.1288e^{-1.4t} + 0.0336e^{-2.1t} \\
 i_{f2} &= 0.5684 - 0.1232e^{-0.7t} - 0.1288e^{-1.4t} + 0.0336e^{-2.1t} \\
 r_{m2} &= 0, & r_{f2} &= 0 \\
 & & & & & & & \vdots
 \end{aligned}$$

3.2.5. Solution of the General Model with the Absence of Effective Counseling and ART

Consider (2.17)-(2.20) with parameters

$$b = 0.5, \quad \alpha_0 = 0.2, \quad \sigma_f = 0.3, \quad \sigma_m = 0.3, \quad c_m = 3, \quad c_f = 3, \quad c_m^* = 2, \quad c_f^* = 2$$

$$b_0 = b + \alpha_0 = 0.7, \quad \beta_f = 0.35, \quad \beta_m = 0.35, \quad \beta_f^* = 0.2, \quad \beta_m^* = 0.2$$

$$T = 10, \quad k = 0.1, \quad \alpha = \alpha_0 e^{-kT} = 0.07$$

and initial conditions

$$i_m(0) = 0.35, \quad i_f(0) = 0.35, \quad r_m(0) = 0, \quad r_f(0) = 0$$

$$i'_m = (1.05i_f + 0.4r_f)(1 - i_m - r_m) - i_m + 0.07i_m r_m + 0.07i_m^2, \quad i_m(0) = 0.35$$

$$i'_f = (1.05i_m + 0.4r_m)(1 - i_f - r_f) - i_f + 0.07i_f r_f + 0.07i_f^2, \quad i_f(0) = 0.35$$

which yields

$$r'_m = 0.3i_m - 0.57r_m + 0.2i_m r_m + 0.07r_m^2, \quad r_m(0) = 0$$

$$r'_f = 0.3i_f - 0.57r_f + 0.2i_f r_f + 0.07r_f^2, \quad r_f(0) = 0$$

The correction functional for (4.24) reads

$$\begin{aligned} i_{m,n+1} &= i_{m,n} + \int_0^t \lambda_1 \{i'_{m,n} - (1.05i_f + 0.4r_f)(1 - i_m - r_m) + i_m - 0.07i_m r_m - 0.07i_m^2\} ds, \\ i_{f,n+1} &= i_{f,n} + \int_0^t \lambda_2 \{i'_{f,n} - (1.05i_m + 0.4r_m)(1 - i_f - r_f) + i_f - 0.07i_f r_f - 0.07i_f^2\} ds, \\ r_{m,n+1} &= r_{m,n} + \int_0^t \lambda_3 \{r'_{m,n} - 0.3i_m + 0.57r_m - 0.2i_m r_m - 0.07r_m^2\} ds \\ r_{f,n+1} &= r_{f,n} + \int_0^t \lambda_4 \{r'_{f,n} - 0.3i_f + 0.57r_f - 0.2i_f r_f - 0.07r_f^2\} ds \end{aligned} \tag{3.25}$$

Integration gives

$$\begin{aligned} i_{m,n+1} &= (1 + \lambda_1)i_{m,n} + \int_0^t (\lambda'_1 - \lambda_1)i_{m,n}(s) ds, \\ i_{f,n+1} &= (1 + \lambda_2)i_{f,n} + \int_0^t (\lambda'_2 - \lambda_2)i_{f,n}(s) ds, \\ r_{m,n+1} &= (1 + \lambda_3)r_{m,n} + \int_0^t (\lambda'_3 - 0.57\lambda_3)r_{m,n}(s) ds \\ r_{f,n+1} &= (1 + \lambda_4)r_{f,n} + \int_0^t (\lambda'_4 - 0.57\lambda_4)r_{f,n}(s) ds \end{aligned}$$

Taking variational derivative on both sides gives

$$\begin{aligned} \delta i_{m,n+1} &= (1 + \lambda_1)\delta i_{m,n} + \int_0^t (\lambda'_1 - \lambda_1)\delta i_{m,n}(s) ds = 0, \\ \delta i_{f,n+1} &= (1 + \lambda_2)\delta i_{f,n} + \int_0^t (\lambda'_2 - \lambda_2)\delta i_{f,n}(s) ds = 0, \\ \delta r_{m,n+1} &= (1 + \lambda_3)\delta r_{m,n} + \int_0^t (\lambda'_3 - 0.57\lambda_3)\delta r_{m,n}(s) ds = 0 \\ \delta r_{f,n+1} &= (1 + \lambda_4)\delta r_{f,n} + \int_0^t (\lambda'_4 - 0.57\lambda_4)\delta r_{f,n}(s) ds = 0 \end{aligned}$$

This yields the stationary conditions

$$\begin{aligned} 1 + \lambda_1(s) &= 0, & \lambda'_1|_{s=t} - \lambda_1(s) &= 0, \\ 1 + \lambda_2(s) &= 0, & \lambda'_2|_{s=t} - \lambda_2(s) &= 0, \\ 1 + \lambda_3(s) &= 0 & \lambda'_3|_{s=t} - 0.57\lambda_3(s) &= 0 \\ 1 + \lambda_4(s) &= 0 & \lambda'_4|_{s=t} - 0.57\lambda_4(s) &= 0 \end{aligned}$$

As a result, we find

$$\lambda_1(s) = \lambda_2(s) = -e^{(s-t)}, \quad \lambda_3(s) = \lambda_4(s) = -e^{0.57(s-t)} \quad (3.26)$$

Substituting (3.26) into (3.25) gives

$$\begin{aligned} i_{m,n+1} &= i_{m,n} - \int_0^t e^{(s-t)} \{i'_{m,n} - (1.05i_f + 0.4r_f)(1 - i_m - r_m) + i_m - 0.07i_m r_m - 0.07i_m^2\} ds, \\ i_{f,n+1} &= i_{f,n} - \int_0^t e^{(s-t)} \{i'_{f,n} - (1.05i_m + 0.4r_m)(1 - i_f - r_f) + i_f - 0.07i_f r_f - 0.07i_f^2\} ds, \\ r_{m,n+1} &= r_{m,n} - \int_0^t e^{0.57(s-t)} \{r'_{m,n} - 0.3i_m + 0.57r_m - 0.2i_m r_m - 0.07r_m^2\} ds \\ r_{f,n+1} &= r_{f,n} - \int_0^t e^{0.57(s-t)} \{r'_{f,n} - 0.3i_f + 0.57r_f - 0.2i_f r_f - 0.07r_f^2\} ds \end{aligned} \quad (3.27)$$

Accordingly, we obtain the following series approximations:

$$\begin{aligned} i_{m0} &= 0.35, \quad i_{f0} = 0.35, \quad r_{m0} = 0, \quad r_{f0} = 0 \\ i_{m1} &= 0.24745 + 0.10255e^{-t} \\ i_{f1} &= 0.24745 + 0.10255e^{-t} \\ r_{m1} &= 0.1842105263 - 0.1842105263e^{-0.57t} \\ r_{f1} &= 0.1842105263 - 0.1842105263e^{-0.57t} \\ i_{m2} &= 0.1970220361 + 0.0822992193e^{-t} + 0.06037257212e^{-2t} + 0.009702290864e^{-1.57t} \\ &\quad + 0.01030617245e^{-3t} - 0.02606928947e^{-2.57t} + 0.0135734072e^{-2.14t} + 0.01636699861e^{-0.57t} \\ &\quad - 0.0135734072e^{-1.14t} \\ i_{f2} &= 0.1970220361 + 0.0822992193e^{-t} + 0.06037257212e^{-2t} + 0.009702290864e^{-1.57t} + 0.01030617245e^{-3t} \\ &\quad - 0.02606928947e^{-2.57t} + 0.0135734072e^{-2.14t} + 0.01636699861e^{-0.57t} \\ &\quad - 0.0135734072e^{-1.14t} \\ r_{m2} &= 0.1503981144 + 0.0094838655e^{-0.57t} - 0.0672303785e^{-1.57t} - 0.1557147057e^{-1.14t} \\ &\quad + 0.006628347183e^{-2.14t} - 0.00416727414e^{-1.71t} + 0.06060203139e^{-t} \\ r_{f2} &= 0.1503981144 + 0.0094838655e^{-0.57t} - 0.0672303785e^{-1.57t} - 0.1557147057e^{-1.14t} \\ &\quad + 0.006628347183e^{-2.14t} - 0.00416727414e^{-1.71t} + 0.06060203139e^{-t}. \end{aligned}$$

3.2.6. Solution of the General Model with the Presence of very Effective Counseling and ART

Consider (2.17)-(2.20) with parameters

$$b = 0.5, \quad \alpha_0 = 0.2, \quad \sigma_f = 0.8, \quad \sigma_m = 0.8, \quad c_m = 3, \quad c_f = 3, \quad c_m^* = 1, \quad c_f^* = 1$$

$$b_0 = b + \alpha_0 = 0.7, \quad \beta_f = 0.15, \quad \beta_m = 0.15, \quad \beta_f^* = 0, \quad \beta_m^* = 0$$

$$T = 10, \quad k = 1, \quad \alpha = \alpha_0 e^{-kT} = 0.03$$

and initial conditions $i_m(0) = 0.35$, $i_f(0) = 0.35$, $r_m(0) = 0$, $r_f(0) = 0$ which yields

$$\begin{aligned} i'_m &= 0.45i_f(1 - i_m - r_m) - 1.5i_m + 0.03i_m r_m + 0.03i_m^2, \quad i_m(0) = 0.35 \\ i'_f &= 0.45i_m(1 - i_f - r_f) - 1.5i_f + 0.03i_f r_f + 0.03i_f^2, \quad i_f(0) = 0.35 \\ r'_m &= 0.8i_m - 0.537r_m + 0.2i_m r_m + 0.03r_m^2, \quad r_m(0) = 0 \\ r'_f &= 0.8i_f - 0.537r_f + 0.2i_f r_f + 0.03r_f^2, \quad r_f(0) = 0 \end{aligned} \quad (3.28)$$

The correction functional for (3.28) reads

$$\begin{aligned} i_{m,n+1} &= i_{m,n} + \int_0^t \lambda_1 \{i'_{m,n} - 0.45i_f(1 - i_m - r_m) + 1.5i_m - 0.03i_m r_m - 0.03i_m^2\} ds, \\ i_{f,n+1} &= i_{f,n} + \int_0^t \lambda_2 \{i'_{f,n} - 0.45i_m(1 - i_f - r_f) + 1.5i_f - 0.03i_f r_f - 0.03i_f^2\} ds, \\ r_{m,n+1} &= r_{m,n} + \int_0^t \lambda_3 \{r'_{m,n} - 0.8i_m + 0.537r_m - 0.2i_m r_m - 0.03r_m^2\} ds \\ r_{f,n+1} &= r_{f,n} + \int_0^t \lambda_4 \{r'_{f,n} - 0.8i_f + 0.537r_f - 0.2i_f r_f - 0.03r_f^2\} ds \end{aligned} \quad (3.29)$$

Integration gives

$$\begin{aligned} i_{m,n+1} &= (1 + \lambda_1)i_{m,n} + \int_0^t (\lambda'_1 - 1.5\lambda_1)i_{m,n}(s) ds, \\ i_{f,n+1} &= (1 + \lambda_2)i_{f,n} + \int_0^t (\lambda'_2 - 1.5\lambda_2)i_{f,n}(s) ds, \\ r_{m,n+1} &= (1 + \lambda_3)r_{m,n} + \int_0^t (\lambda'_3 - 0.53\lambda_3)r_{m,n}(s) ds \\ r_{f,n+1} &= (1 + \lambda_4)r_{f,n} + \int_0^t (\lambda'_4 - 0.53\lambda_4)r_{f,n}(s) ds \end{aligned}$$

Taking variational derivative both sides gives

$$\begin{aligned} \delta i_{m,n+1} &= (1 + \lambda_1)\delta i_{m,n} + \int_0^t (\lambda'_1 - 1.5\lambda_1)\delta i_{m,n}(s) ds = 0, \\ \delta i_{f,n+1} &= (1 + \lambda_2)\delta i_{f,n} + \int_0^t (\lambda'_2 - 1.5\lambda_2)\delta i_{f,n}(s) ds = 0, \\ \delta r_{m,n+1} &= (1 + \lambda_3)\delta r_{m,n} + \int_0^t (\lambda'_3 - 0.53\lambda_3)\delta r_{m,n}(s) ds = 0 \\ \delta r_{f,n+1} &= (1 + \lambda_4)\delta r_{f,n} + \int_0^t (\lambda'_4 - 0.53\lambda_4)\delta r_{f,n}(s) ds = 0 \end{aligned}$$

This yields the stationary conditions

$$\begin{aligned} 1 + \lambda_1(s) &= 0, & \lambda'_1|_{s=t} - 1.5\lambda_1(s) &= 0, \\ 1 + \lambda_2(s) &= 0, & \lambda'_2|_{s=t} - 1.5\lambda_2(s) &= 0, \\ 1 + \lambda_3(s) &= 0 & \lambda'_3|_{s=t} - 0.53\lambda_3(s) &= 0 \\ 1 + \lambda_4(s) &= 0 & \lambda'_4|_{s=t} - 0.53\lambda_4(s) &= 0 \end{aligned}$$

As a result, we find

$$\lambda_1(s) = \lambda_2(s) = -e^{1.5(s-t)}, \lambda_3(s) = \lambda_4(s) = -e^{0.53(s-t)} \tag{3.30}$$

Substituting (2.30) into (2.29) gives

$$\begin{aligned} i_{m,n+1} &= i_{m,n} - \int_0^t e^{1.5(s-t)} \{i'_{m,n} - 0.45i_f(1 - i_m - r_m) + 1.5i_m - 0.03i_m r_m - 0.03i_m^2\} ds, \\ i_{f,n+1} &= i_{f,n} - \int_0^t e^{1.5(s-t)} \{i'_{f,n} - 0.45i_m(1 - i_f - r_f) + 1.5i_f - 0.03i_f r_f - 0.03i_f^2\} ds, \\ r_{m,n+1} &= r_{m,n} - \int_0^t e^{0.53(s-t)} \{r'_{m,n} - 0.8i_m + 0.537r_m - 0.2i_m r_m - 0.03r_m^2\} ds \\ r_{f,n+1} &= r_{f,n} - \int_0^t e^{0.53(s-t)} \{r'_{f,n} - 0.8i_f + 0.537r_f - 0.2i_f r_f - 0.03r_f^2\} ds \end{aligned} \tag{3.31}$$

Accordingly, we obtain the following series approximations:

$$\begin{aligned} i_{m0} &= 0.35, & i_{f0} &= 0.35, & r_{m0} &= 0, & r_{f0} &= 0 \\ i_{m1} &= 0.0707 + 0.2793e^{-1.5t} \\ i_{f1} &= 0.0707 + 0.2793e^{-1.5t} \\ r_{m1} &= 0.5283018868 - 0.5283018868e^{-0.53t} \\ r_{f1} &= 0.5283018868 - 0.5283018868e^{-0.53t} \\ i_{m2} &= 0.00935215865 + 0.092764475e^{-1.5t} + 0.2260409892e^{-3t} + 0.0308570566e^{-2.03t} + 0.0218423772e^{-4.5t} \\ &\quad - 0.04131532076e^{-3.53t} + 0.01045826415e^{-0.53t} \\ i_{f2} &= 0.00935215865 + 0.092764475e^{-1.5t} + 0.2260409892e^{-3t} + 0.0308570566e^{-2.03t} + 0.0218423772e^{-4.5t} \\ &\quad - 0.04131532076e^{-3.53t} + 0.01045826415e^{-0.53t} \\ r_{m2} &= 0.136099532 + 0.3460006851e^{-0.53t} - 0.5329469562e^{-2.03t} - 0.4668123619e^{-1.06t} \\ &\quad + 0.05568102528e^{-2.03t} - 0.01579827643e^{-1.59t} + 0.4772659309e^{-1.5t} \\ r_{f2} &= 0.136099532 + 0.3460006851e^{-0.53t} - 0.5329469562e^{-2.03t} - 0.4668123619e^{-1.06t} \\ &\quad + 0.05568102528e^{-2.03t} - 0.01579827643e^{-1.59t} + 0.4772659309e^{-1.5t} \end{aligned}$$

3.3. Graphical Illustration of Results

In this section, we observe the behaviour of the sub-models and general model graphically by modifying key parameter values. Graphical results are obtained using MATLAB version 7.5.

The graphical results are meant to study the following situations:

I. The prevalence of infection in the sub-model without ART.

II. The prevalence of infection in the sub-model with only infected males receiving ART.

III. The prevalence of infection in the sub-model with only infected females receiving ART.

IV. The prevalence of infection in the general model with the absence of counseling and ART.

V. The prevalence of infection in the general model with the absence of effective counseling and ART.

VI. The prevalence of infection in the general model with the presence of very effective counseling and ART.

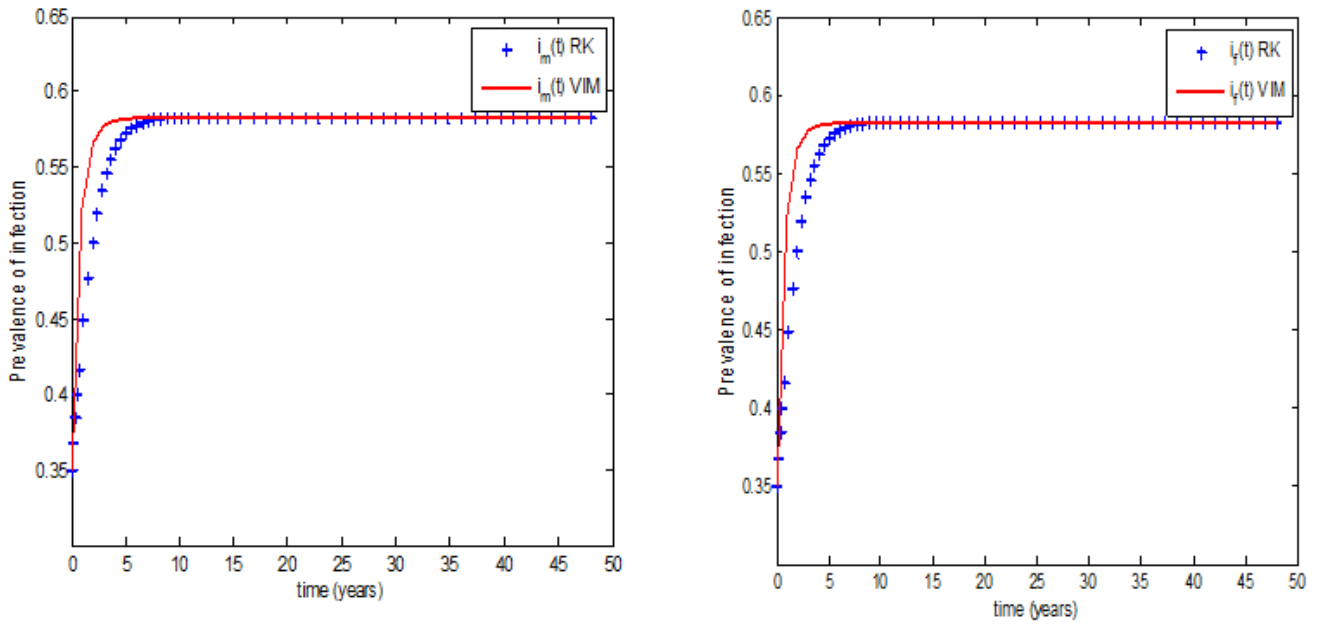


Figure 2. The prevalence of infection in the sub-model without ART (Experiment One)

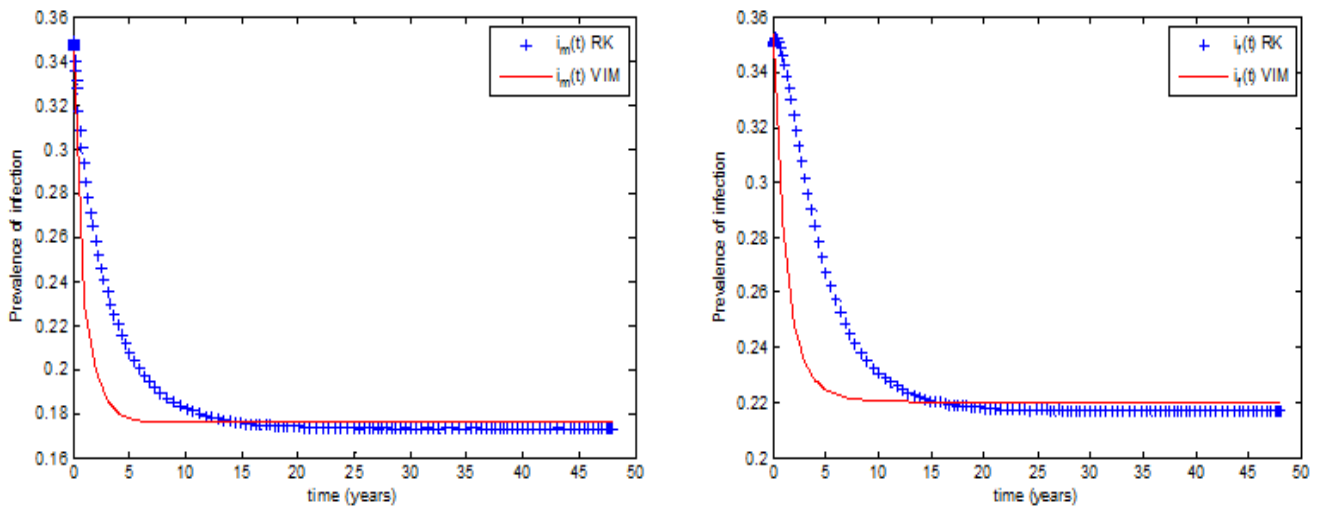


Figure 3. The prevalence of infection in the sub-model with only infected males receiving ART (Experiment Two)

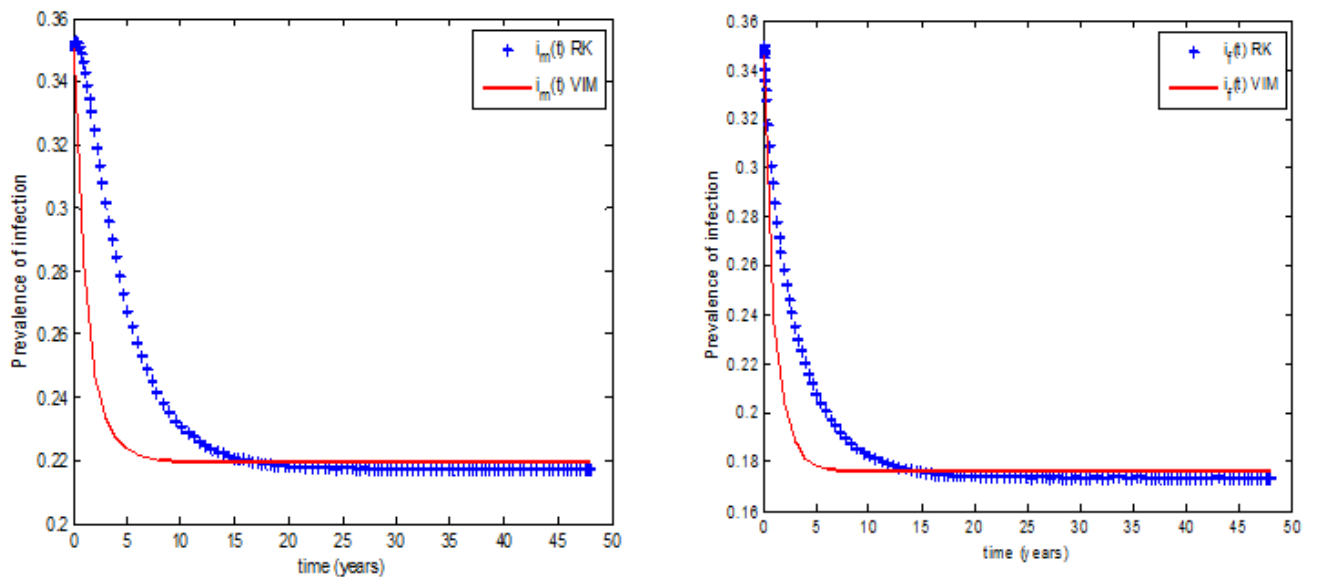


Figure 4. The prevalence of infection in the sub-model with only infected females receiving ART (Experiment Three)

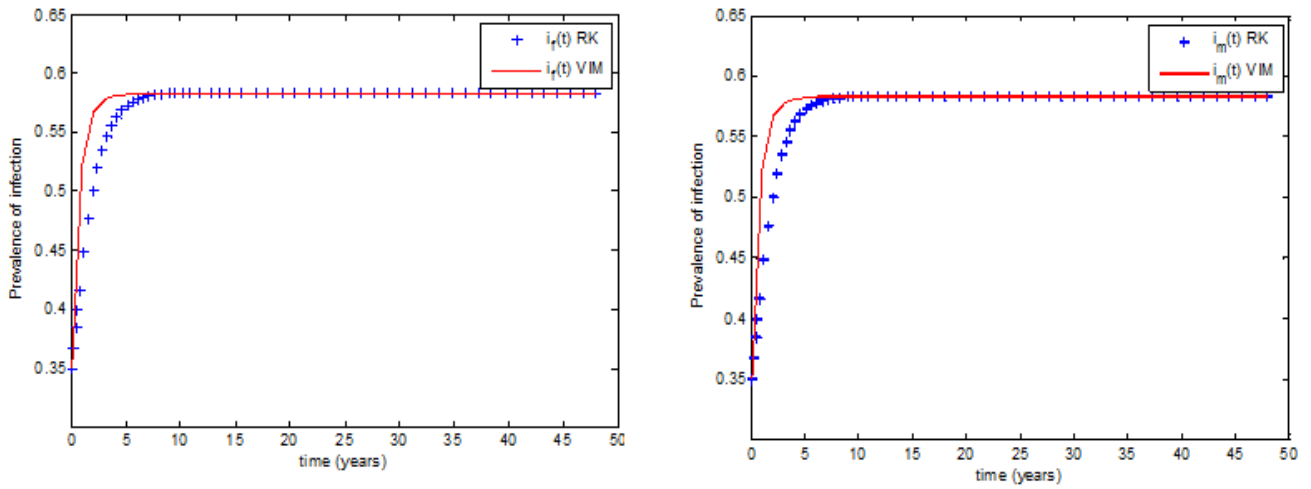


Figure 5. The prevalence of infection in the general model with the absence of counseling and ART (Experiment Four)

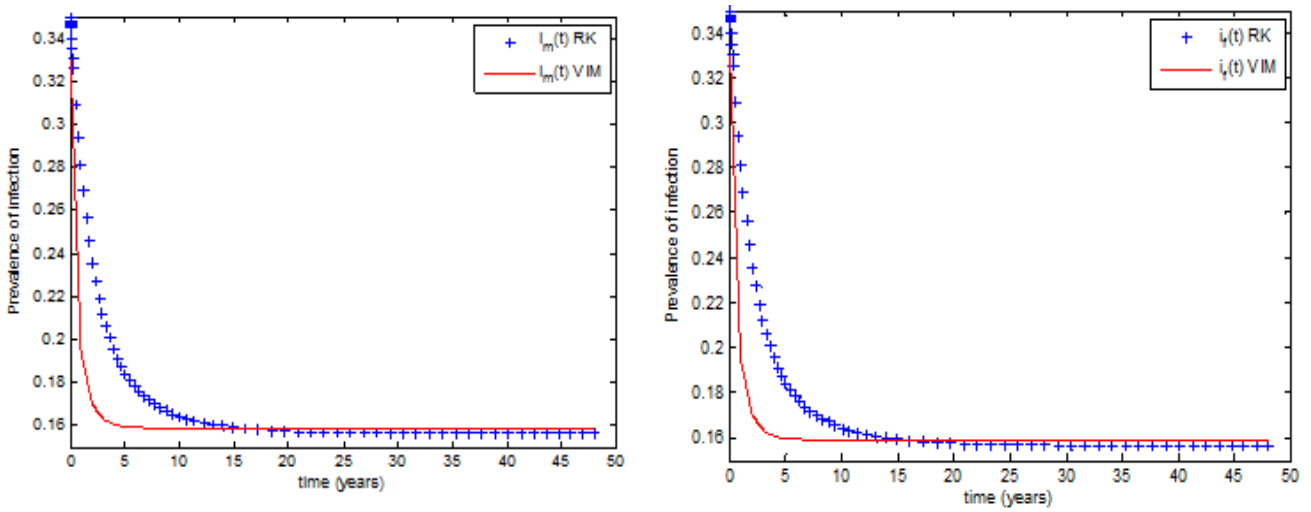


Figure 6. The prevalence of infection in the general model with the absence of effective counseling and ART (Experiment Five)

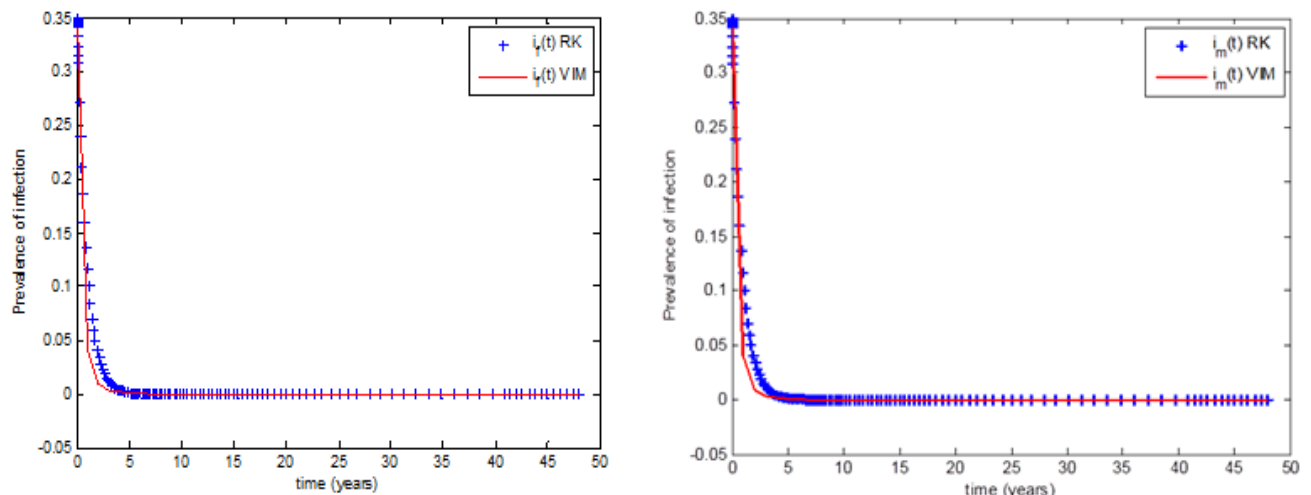


Figure 7. The prevalence of infection in the general model with the presence of very effective counseling and ART (Experiment Six)

3.4. Discussion

Therefore, based on the results obtained in this finding the model defined in Section 2.2.3. The analysis of the model is being carried out. The graphs plotted using these results are shown in Figure 2-7 of Section 3.3 above.

The model formulated by Kimbir *et al.* (2012), was solved using the fourth-order Runge Kutta method and the results were obtained. We solve the model using the variational iteration method and make comparisons as necessary to these results to ensure consistency.

3.5. Discussion of the General Model Analysis

The result from the existence and stability analysis of the equilibrium state of the general model shows that the disease-free equilibrium is locally and asymptotically stable (LAS) since all the eigenvalues of the model equation are less than zero.

3.6. Discussion of the Result of the Model Equations

3.6.1. Experiment One

In this case, we study the prevalence of infection in the Sub-model without ART. The result displayed in Figure 2, shows high prevalence of infection for both males and females. We note that the graphs of variational iteration method (VIM) and Runge Kutta method (RKM) coincide for most point except the difference between 1 and 8 years.

3.6.2. Experiment Two

In this case, we study the prevalence of infection in the Sub-model with only infected males receiving ART. The result displayed in Figure 3, shows a decrease in the prevalence of infection for males which resulted to decrease in the prevalence of infection for females. This however does not achieve the desire result of eradicating the infection. This is due to the very low efficacy level of ART ($k = 0.1$ and $\sigma_m = \sigma_f = 0.4$) and high number of sexual partners ($c_m^* = c_f^* = 2$). We note here that the graphs of variational iteration method (VIM) and Runge kutta method (RKM) almost coincide except the difference between 1 and 14 years.

3.6.3. Experiment Three

In this case, we study the prevalence of infection in the Sub-model with only infected females receiving ART. The result displayed in Figure 4, also shows a decrease in the prevalence of infection for females which resulted to decrease in the prevalence of infection for males. This does not, however, achieve eradication (like Experiment 2) as compared to experiment 6 above. We note that the graphs of the variational iteration method (VIM) and Runge Kutta method (RKM) appeared to be one except the difference between 1 and 11 years.

3.6.4. Experiment Four

In this case, we study the prevalence of infection in the general model with the absence of counseling and ART. The result displayed in Figure 6, shows high prevalence on infection for both males and females (like Experiment 1). We note a similar behaviour in the graphs of the variational iteration method and Runge Kutta method (RKM) as compared to Experiment 1 above. We note here also that the graphs of the variational iteration method (VIM) and Runge Kutta method (RKM) appear to be one except the difference between 1 and 8 years.

3.6.5. Experiment Five

In this case, we study the prevalence of infection in the general model with the absence of effective counseling and ART. The result displayed in Figure 6, shows

decrease in the prevalence of infection. This does not, however, achieve eradication (like Experiment 2 and Experiment 3) as compared to Experiment 6 above. This is due to the increase in the number of sexual partners and transmission rates and low efficacy level of ART. The plots of the variational iteration method and that of Runge Kutta method differ at 1 and 14 years.

3.6.6. Experiment Six

Here, we investigate the prevalence of infection in the general model with the presence of very effective counseling and ART. The result displayed in Figure 7, shows a fast decrease in the prevalence of infection. This is due to a decrease in number of sexual partners ($c_m^* = c_f^* = 1$) and high efficacy level of ART ($k = 1$ and $\sigma_m = \sigma_f = 0.8$). With this, HIV/AIDS can be eradicated completely from heterosexual populations in a finite time. We note that the graphs of variational iteration method (VIM) and Runge Kutta method (RKM) coincide for most points except difference between 2 and 3 years.

4. Conclusion

In this study, the existence of the equilibrium state of the general model investigated and its stability were analyzed. The result in the study revealed the use of ART on infectives shown to be locally and asymptotically stable. The solutions of the sub-models and the general model were obtained through the use of variational iteration method. The results obtained by the variational iteration method were continuous and can be used for further analysis such as differentiation and integration and can also be used to compute prevalence rate of the mesh points. The results of the variational iteration method were also compared to that of the fourth- order Runge Kutta method which affirms that the methods were effective and easy to implement. It was observed that for effective counseling and ART to lead eradication, it is therefore necessary that the same proportion of males and females should be involved in ART.

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Conflict of Interest

The authors declare that there are no conflicts of interest.

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