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Properties of a Certain Class of Meromorphic Analytic Functions Defined by a Linear Operator

Ajai P.T.[1,3,*], Moses B.O.[2], Ihedioha S.A.[3]

[1]Department of Mathematics, University of The Gambia, Banjul, The Gambia
[2]Department of Mathematics, American University of Nigeria, Yola, Adamawa, Nigeria
[3]Department of Mathematics, Plateau state University, Bokkos, Nigeria
*Corresponding author: ajaiterwase@utg.edu.gm

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Abstract In this present paper, we introduced and characterized a new class of meromorphic univalent functions associated with polylogarithm by investigating; coefficient inequality, convolutions property, integral means and other properties of the class.

Keywords: *analytic function, differential operator, Hadamard product, univalent functions*

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1. Introduction and Definitions

Let $\sum$ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=2}^{\infty}a_k z^{k-1} \quad (1.1)$$

Which are analytic in the unit disk $U=\{z:|z|<1=U\backslash\{0\}\}$. Having a simple pole at the origin with residue 1. Furthermore, let $\sum_\alpha$, $\sum^*(\alpha)$ and $\sum_k$, $0\le\alpha<1$ denotes the subclasses of $\sum$ which are univalent, meromorphically starlike and convex respectively.

Definition 1

Analytically, a function of the form (1.1) is in $\sum^*(\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{-zf'(z)}{f(z)}\right\}>\alpha, z\in U, 0\le\alpha<1. \quad (1.2)$$

Definition 2

Similarly, $f\in\sum_k(\alpha)$. If and only if f is of the form (1.1) and satisfies

$$\operatorname{Re}\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha, z\in U, 0\le\alpha<1. \quad (1.3)$$

Definition 3

For $c\in\mathbb{N}$, the set of natural numbers with $c\ge 2$, an absolutely convergent series defined as

$$Li_c(z)=\sum_{k=1}^{\infty}\frac{1}{(k+1)^c}z^k. \quad (1.4)$$

Is known as the polylogarithm. This class of functions was invented by Liebniz and Bernouli [1]. For more works on polylogarithm and meromorphic functions see [2-7].

We state here a linear operator derived as follow;

Let $\Psi_c f(z):\sum\to\sum$ which is defined by the following Hadamard product by $\Psi_c f(z)=\xi_c(z)*f(z)$ Where

$$\xi_c(z)=z^{-2}Li_c(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{1}{(k+1)^c}z^k. \quad (1.5)$$

Define $D_c f(z):\sum\to\sum$ as

$$D_c f(z)=\left\{\Psi_c f(z)-\frac{1}{2^c}a_1\right\}. \quad (1.6)$$

Definition 4

Let $f(z)$ be defined as in (1.1) and $D_c f(z)$ as stated in (1.6) then the function $f(z)$ then the function $f(z)$ in (1.1) is said to be in class $\sum_c(\lambda)$ if the following geometric condition are satisfy;

$$\Re\left\{\frac{-z(D_c f(z))}{D_c f(z)}\right\}>\lambda, 0\le\lambda<1 \quad (1.7)$$

Using subordination we write (1.7) as

$$\left|\frac{1+\frac{z(D_c f(z))'}{D_c f(z)}}{1-2\lambda-\frac{z(D_c f(z))'}{D_c f(z)}}\right|<1, z\in U \qquad (1.8)$$

Where $D_c f(z)$ is as defined in (1.6)

2. Coefficient Inequality

Theorem 2.1

Let $f(z)$ of the form (1.1) a function $f(z)$ is said to be in the class $\sum_c(\lambda)$ iff the following bound is satisfy:

$$\sum_{k=2}^{\infty}\frac{k+\lambda-1}{(k+1)^c}a_k \le \lambda \qquad (2.1)$$

Proof

Assume that (2.1) holds true then from (1.8) we have

$$\left|\frac{z(D_c f(z))'+D_c f(z)}{(1-2\lambda)D_c f(z)-zD_c f(z)'}\right|$$

$$=\left|\frac{k\sum_{k=2}^{\infty}\frac{a_k}{(k+1)^c}z^{k-1}}{-2\lambda\frac{1}{z}+\sum_{k=2}^{\infty}\frac{2(1-\lambda)}{(k+1)^c}a_k z^{k-1}}\right|$$

$$\le\left|\frac{k\sum_{k=2}^{\infty}\frac{a_k}{(k+1)^c}}{-2\lambda\frac{1}{z}+\sum_{k=2}^{\infty}\frac{2(1-\lambda)}{(k+1)^c}a_k}\right|\le 1.$$

Proving (2.1) Conversely, suppose $f(z)\in\sum_c(\lambda)$.

We have to show that condition (2.1) is true. Thus we have

$$\left|\frac{z(D_c f(z))'+D_c f(z)}{(1-2\lambda)D_c f(z)-zD_c f(z)'}\right|\le 1, \qquad (2.2)$$

Which is equivalent to

$$\left|\frac{z(D_c f(z))'+D_c f(z)}{(1-2\lambda)D_c f(z)-zD_c f(z)'}\right|$$

$$=\left|\frac{k\sum_{k=2}^{\infty}\frac{a_k}{(k+1)^c}z^{k-1}}{-2\lambda\frac{1}{z}+\sum_{k=2}^{\infty}\frac{2(1-\lambda)}{(k+1)^c}a_k z^{k-1}}\right|<1.$$

Notice that since $\Re(z)<|z|$ we similarly have

$$\Re\left\{\frac{k\sum_{k=2}^{\infty}\frac{a_k}{(k+1)^c}z^{k-1}}{2\lambda+\sum_{k=2}^{\infty}\frac{2(1-\lambda)}{(k+1)^c}a_k z^{k-1}}\right\}<1 \qquad (2.3)$$

We choose the value z on the real axis and letting $z\to 1^-$, we have

$$\left\{\frac{k\sum_{k=2}^{\infty}\frac{a_k}{(k+1)^c}z^{k-1}}{2\lambda+\sum_{k=2}^{\infty}\frac{2(1-\lambda)}{(k+1)^c}a_k z^{k-1}}\right\}<1 \qquad (2.4)$$

Which proves our assertion. The result is sharp here for the function;

$$f(z)=\frac{1}{z}+\frac{(k+1)^c}{k-(1-2\lambda)}a_k z^{k-1}. \qquad (2.5)$$

Theorem 2.2

The class is closed under convex combination.

Let $f_1(z), f_2(z)\in\sum_c(\lambda)$ then for $0\le\tau<1$, then we have $(1-\tau)f_1(z)+\tau f_2(z)\in\sum_c(\lambda)$.

Proof

By hypothesis $f_1(z)=\frac{1}{z}+\sum_{k=2}^{\infty}a_k z^{k-1}$, and

$$f_2(z)=\frac{1}{z}+\sum_{k=2}^{\infty}b_k z^{k-1}.$$

Then

$$(1-\tau)f_1(z)+\tau f_2(z)=\frac{1}{z}+\sum_{k=2}^{\infty}\left[(1-\tau)a_k+\tau b_k\right]kz^{k-1}.$$

Thus we have from (2.1) the following

$$\sum_{k=2}^{\infty}\frac{k+\lambda-1}{(k+1)^c}\left[(1-\tau)a_k+\tau b_k\right]$$
$$=\sum_{k=2}^{\infty}\frac{k+\lambda-1}{(k+1)^c}(1-\tau)a_k+\sum_{k=2}^{\infty}\frac{k+\lambda-1}{(k+1)^c}\tau b_k$$
$$\le(1-\tau)\lambda+\tau\lambda=\lambda.$$

This complete our proof.

3. Integral Means Inequalities

Let $f(z)$ and $g(z)$ be analytic in U, $f(z)$ is said to be subordinate to $g(z)$ written as

$$f(z)\prec g(z), z\in U. \qquad (3.1)$$

If there exists a Schwarz function $w(z)$ which is analytic in U with $w(0)=0, |w(z)|<1$, $z\in U$ such that $f(z)=g(w(z))$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence, see [8] $f(z)\prec g(z)\Leftrightarrow f(0)=g(0)$ and $f(U)\subset g(U)$.

Theorem 3.1 [9]

If f(g) and g(z) are analytic in U with $f(z)\prec g(z)$, then for $\mu>0$, and $z=re^{i\theta}$, $0<r<1$. Then

$$\int_0^{2\pi}|f(z)|^{\mu}d\theta \le \int_0^{2\pi}|g(z)|^{\mu}d\theta.$$

Theorem 3.2

Let $f(z)\in\sum_c(\lambda)$ and $f_k(z)$ be defined by $f_k(z)=\frac{1}{z}+\frac{\lambda(k+1)^c}{k+\lambda-1}z^{k-1}, k=2,3...$ if there exists w(z) such that

$$w^{k+1}(z)=\frac{\lambda(k+1)^c}{k+\lambda-1}\sum_{k=1}^{\infty}a_k z^{k+1}, k=2,3... \qquad (3.2)$$

$z=re^{i\theta}$ and $0\le r<1$. Then

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^r d\theta \le \int_0^{2\pi}\left|f_k(re^{i\theta})\right|^r d\theta, r>0.$$

Proof

It is obvious that

$$\int_0^{2\pi}\left|1+\sum_{k=2}^{\infty}a_k z^{k-1}\right|^r d\theta \le \int_0^{2\pi}\left|1+\frac{\lambda(k+1)^c}{k+\lambda-1}z^{k-1}\right|^r d\theta, r>0$$

Using theorem 3.1 we have to show that

$$1+\sum_{k=2}^{\infty}a_k z^{k-1} \prec 1+\frac{\lambda(k+1)^c}{k+\lambda-1}z^{k-1} \qquad (3.4)$$

Suppose we set $w^k(z)=\frac{\lambda(k+1)^c}{k+\lambda-1}\sum_{k=1}^{\infty}a_k z^k$. Then we have $1+\sum_{k=2}^{\infty}a_k z^{k-1}\le 1+\frac{\lambda(k+1)^c}{k+\lambda-1}z^{k-1}$

$$1+\sum_{k=1}^{\infty}a_k z^{k-1}=1+\frac{\lambda(k+1)^c}{k+\lambda-1}|w(z)|^k.$$

Notice that $w(0)=0$ and from theorem 2.1 we can write

$$w^k(z)=\left|\frac{\lambda}{(k+\lambda-1)(k+1)^c}\sum_{k=1}^{\infty}a_k z^k\right|$$
$$\le\frac{\lambda}{(k+\lambda-1)(k+1)^c}\sum_{k=1}^{\infty}|a_k|$$
$$\le|z|<1.$$

This proves our theorem.

4. Convolution Property

Let $f(z)$, $g(z)\in\sum_c(\lambda)$ and

$f(z)=\sum_{k=2}^{\infty}a_k z^{k-1}, g(z)=\sum_{k=2}^{\infty}b_k z^{k-1}$ Robbinson [10]

has shown that $f(z)*g(z)=\frac{1}{z}+\sum_{k=2}^{\infty}a_k b_k z^{k-1}$

is also in $\sum_c(\lambda)$.

Theorem 4.1

Suppose $f(z)$, $g(z)\in\sum_c(\lambda)$ then the Hadamard product or convolution of the functions f and g belongs to the class $\sum_c(\lambda_1)$. Where $\lambda_1\ge\frac{1-k}{1-(k+1)^c}$.

Proof.

Since $f(z), g(z)\in\sum_c(\lambda)$, from theorem 2.1 we have

$$\sum_{k=2}^{\infty}\frac{k+\lambda-1}{\lambda(k+1)^c}a_k\le 1 \text{ and } \sum_{k=2}^{\infty}\frac{k+\lambda-1}{\lambda(k+1)^c}b_k\le 1.$$

We need to find the largest λ_1 $\sum_{k=2}^{\infty}\frac{k+\lambda-1}{\lambda(k+1)^c}a_k b_k\le 1$, by Cauchy-Schwarz inequality, we have

$$\sum_{k=2}^{\infty}\frac{k+\lambda-1}{\lambda(k+1)^c}\sqrt{a_k b_k}\le 1. \qquad (3.5)$$

Thus it suffices to show that

$$\frac{k+\lambda_1-1}{\lambda_1(k+1)^c}a_k b_k\le\frac{k+\lambda-1}{\lambda_1(k+1)^c}\sqrt{a_k b_k}$$

Which is equivalent to

But from (3.5) we have

$$\sqrt{a_k b_k}\le\frac{(k+1)^c\lambda}{k+\lambda-1}$$
$$\frac{(k+1)^c}{k+\lambda-1}\le\frac{(k+\lambda_1-1)}{(k+\lambda-1)\lambda_1}$$

The above simplify to $\lambda_1\ge\frac{1-k}{1-(k+1)^c}$. This proves our result.

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