American Journal of Applied Mathematics and Statistics, 2018, Vol. 6, No. 6, 239-243
Available online at http://pubs.sciepub.com/ajams/6/6/4
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DOI:10.12691/ajams-6-6-4

A Class of Weighted Laplace Distribution

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Received August 17, 2018; Revised October 25, 2018; Accepted November 13, 2018

Abstract The weighted Laplace model is proposed following the method of Azzalini (1985). Basic properties of the distribution including moments, generating function, hazard rate function and estimation of parameters have been studied.

Keywords: *Laplace distribution, hazard function, moments, likelihood function*

Cite This Article: Yasser M. Amer, “A Class of Weighted Laplace Distribution.” *American Journal of Applied Mathematics and Statistics*, vol. 6, no. 6 (2018): 239-243. doi: 10.12691/ajams-6-6-4.

1. Introduction

Azzalini [1] was the first to propose the skew-normal distribution to incorporate (shape/skewness) parameter to a normal distribution depending on a weighted function denoted by $F(\alpha X)$ where α is a shape parameter. Azzalini’s proposition was followed by extensive work that aimed to introduce ‘shape’ parameters to some symmetric distributions, for instance skew-t, Skew-Cauchy, Skew-Laplace, and skew-logistic. In general, skew-symmetric distributions have been defined and several of their properties and inference procedures have been discussed, see for example, Arnold and Beaver [2], Gupta and Kundu [3] and the recent study by Genton [4]. Arnold and Beaver [5] provided a nice explanation of Azzalini's skew-normal distribution as a hidden truncation model, although the same explanation may not be true for other skewed distributions.

The Laplace distribution, its name came after Pierre-Simon Laplace (1749-1827) obtained the likelihood of the Laplace distribution and found it is maximized when the location parameter is set to be the median. Sometimes the distribution is called the double exponential distribution, because it can be thought of as two exponential distributions (with an additional location parameter) spliced together back-to-back, although the term is also sometimes used to refer to the Gumbel distribution.

Up to this day, many studies have been published with extensions and applications of the Laplace distribution. Extensions to a skewed model as well as to a multivariate setting can be found, for example, in Kotz et al. [6] and references therein. Liu and Kozubowski [7] have studied a class of probability distributions on the positive line, which arise when folding the classical Laplace distribution around the origin. Yu and Moyeed [8] and Yu and Zhang [9] have proposed a three-parameter asymmetric Laplace distribution. Cordeiro and Lemonte [10] have proposed the socalled beta Laplace distribution as an extension of the Laplace distribution.

In this study, we will provide a new generalization of Laplace distribution called the Weighted Laplace distribution.

2. The Weighted Laplace Distribution

A method of obtaining weighted distributions from independently identically distributed (i.i.d.) random variables was proposed by Azzalini [1]. This proposed family of distributions used density function of one random variable and distribution function of other random variable. To simplify the idea, suppose two random variables X and Y are i.i.d. random variables with distribution function $F(x)$. Azzalini [1] proposed that a weighted class of density functions can be obtained by using

$$f_X(x)=\frac{1}{p(\alpha X>Y)}g(x)G_Y(x);\alpha>0. \tag{1}$$

Gupta and Kundu [3] used equation (1) to suggest the weighted exponential distribution.

Now when X and Yare i.i.d. as Laplace random variable with parameter β.Then apply

$$g_X(x)=\frac{1}{2\beta}Exp(-\frac{1}{\beta}|x|);\beta>0$$

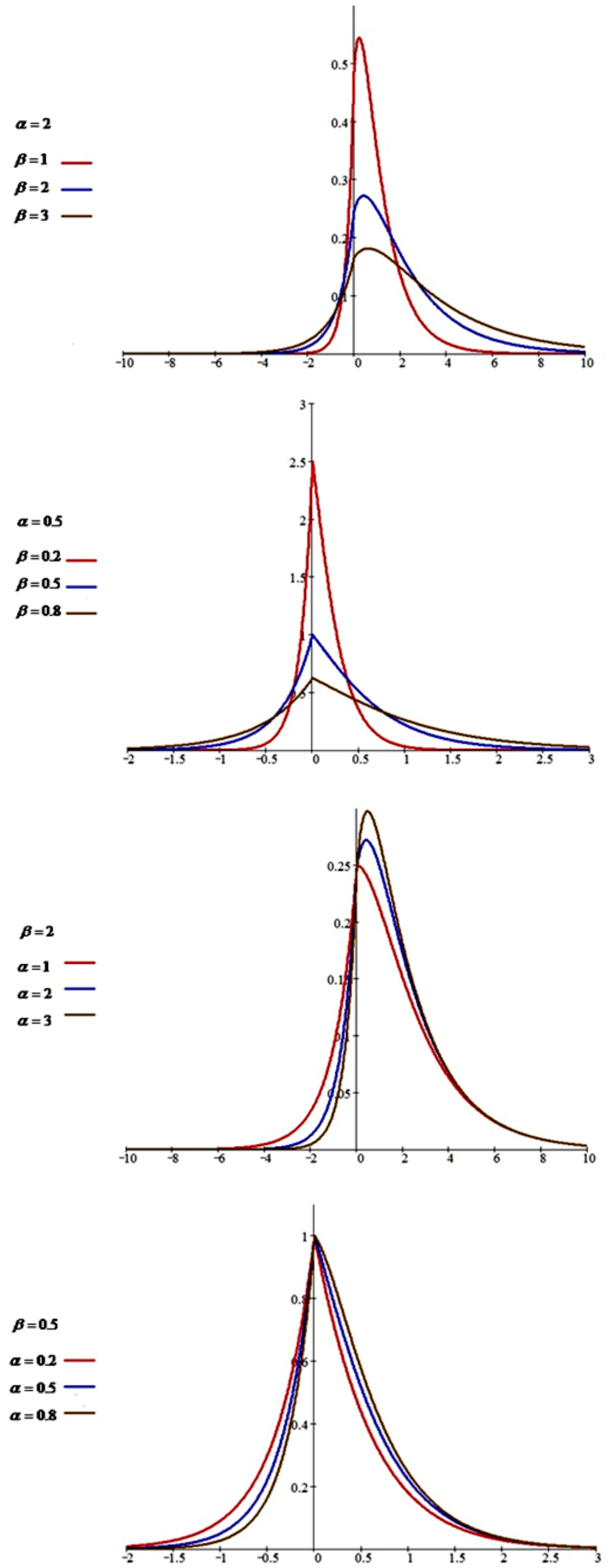
and

$$G_Y(x)=\frac{1}{2}\left\{sgn(x)\left[1-\text{Exp}\left(-\frac{1}{\beta}|x|\right)\right]+1\right\}$$

in (1) we obtain following weighted Laplace distribution (WLD):

$$f_X(x)=\frac{1}{2\beta}Exp\left(-\frac{1}{\beta}|x|\right)\left\{sgn(x)\left[1-Exp\left(-\frac{\alpha}{\beta}|x|\right)\right]+1\right\}. \tag{2}$$

The plots of density function (2) for different choices of β an α are given below:



We now present some common properties of distribution WLD in the following sections.

3. The Statistical Properties of (WLD)

In this section, we present the statistical properties of (WLD) throughout computing the moment generating function, the rth moment, mean, variance, reliability function and hazard function as follow:

3.1. The Cumulative Distribution Function (CDF)

We have defined the weighted Laplace distribution in (2). The Cumulative distribution function of (2) is given as:

$$F_x(X) = \int_{-\infty}^{x} f(w)\,dw \tag{3}$$

At $x < 0$

$$F_x(X) = \int_{-\infty}^{x} \frac{1}{2\beta} Exp\left(\frac{w}{\beta}\right) Exp\left(\frac{\alpha w}{\beta}\right) dw.$$

After simplification, the distribution function is:

$$F_x(X) = \frac{1}{2(\alpha+1)} Exp\left(\frac{(\alpha+1)x}{\beta}\right) x<0. \tag{3}$$

At $x > 0$

$$F_x(X) = \int_{-\infty}^{0} f(w)\,dw + \int_{0}^{x} f(w)\,dw.$$

After simplification, the distribution function is:

$$F_x(X) = \frac{1}{2(\alpha+1)} Exp\left(\frac{-(\alpha+1)x}{\beta}\right) - Exp\left(\frac{-x}{\beta}\right) + 1,\ x>0.$$

Then

$$F_x(X) = \frac{1}{2(\alpha+1)} Exp\left(\frac{-(\alpha+1)|x|}{\beta}\right) - Exp\left(\frac{-Z_x}{\beta}\right) + 1 \tag{4}$$

Where

$$Z_x = \begin{cases} 0 \ if \ x<0 \\ x \ if \ x>0 \end{cases}$$

3.2. The Hazard Rate Function

Using (2) and (4), the hazard rate function of weighted Laplace distribution is:

$$h(X) = \frac{f_X(x)}{1-F_x(X)}$$

$$= \frac{\frac{1}{2\beta} Exp\left(-\frac{1}{\beta}|x|\right)\left\{sgn(x)\left[1-Exp\left(-\frac{\alpha}{\beta}|x|\right)\right]+1\right\}}{Exp\left(\frac{-Z_x}{\beta}\right) - \frac{1}{2(\alpha+1)} Exp\left(\frac{-(\alpha+1)|x|}{\beta}\right)}. \tag{5}$$

3.3. The Moment Generating Function

The moment generating function of density (2) can readily obtain as:

$$M_X(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\beta} Exp\left(-\frac{1}{\beta}|x|\right)\left\{sgn(x)\left[1-Exp\left(-\frac{\alpha}{\beta}|x|\right)\right]+1\right\} dx$$

$$= \int_{-\infty}^{0} \frac{1}{2\beta} e^{tx} e^{\frac{(\alpha+1)x}{\beta}} dx + \int_{0}^{\infty} \frac{1}{2\beta} e^{tx} e^{\frac{-x}{\beta}} \left(2 - e^{\frac{-\alpha x}{\beta}}\right) dx.$$

After simplification, the moment generating function of (WLD) is:

$$M_X(t)=\frac{\alpha^2+2\alpha-t\beta+1}{\left\{(t\beta)^2+(\alpha+1)^2\right\}(1-t\beta)}. \quad (6)$$

Note that

$$M_X(0)=1.$$

Mean and variance of weighted Laplace distribution can be found

$$E(X)=\left.\frac{dM_X(t)}{dt}\right|_{t=0}=\frac{\beta\alpha(\alpha+2)}{(\alpha+1)^2} \quad (7)$$

by using (6).

$$E\left(X^2\right)=\left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0}=\frac{2\beta^2\left\{(\alpha+1)^3-1\right\}}{(\alpha+1)^3} \quad (8)$$

Then the variance of weighted Laplace distribution is

$$V(x)=E\left(X^2\right)-\{E(X)\}^2=\frac{2\alpha\beta^2\left\{2\alpha^2+\alpha+3\right\}}{(\alpha+1)^4}.$$

3.4. The r^{th} Moment

Now let us consider the different moments of the weighted Laplace distribution. Suppose X denote the weighted Laplace distribution random variable with parameter β and, then:

$$\mathrm{E}\left(\mathrm{X^r}\right)=\int_{-\infty}^{\infty}\mathrm{X^r f(x)dx}$$

$$=\int_{-\infty}^{\infty}\mathrm{X^r}\frac{1}{2\beta}Exp\left(-\frac{1}{\beta}|x|\right)\left\{sgn(x)\left[1-Exp\left(-\frac{\alpha}{\beta}|x|\right)\right]+1\right\}d\mathrm{x}$$

$$=\int_{-\infty}^{0}x^r\frac{1}{2\beta}e^{\frac{(\alpha+1)x}{\beta}}dx+\int_{0}^{\infty}x^r\frac{1}{2\beta}e^{-\frac{x}{\beta}}\left\{2-e^{-\frac{\alpha x}{\beta}}\right\}dx$$

After simplification, ther^{th} moment of (TLD) is:

$$E\left(X^r\right)=\Gamma(r+1)\left\{\frac{(-\beta)^r}{2(\alpha+1)^{r+1}}-\frac{\beta^r(1-\lambda)}{2(\alpha+1)^{r+1}}+\beta^r\right\}. \quad (9)$$

Therefore putting $\mathrm{r}=1$, we obtain the mean as

$$E(x)=\frac{\beta\alpha(\alpha+2)}{(\alpha+1)^2}$$

and putting $\mathrm{r}=2$ we obtain the second moment as

$$E\left(x^2\right)=\frac{2\beta^2\left((\alpha+1)^3-1\right)}{(\alpha+1)^3}.$$

These results are the same results previously obtained in (7) and (8) and can also be access to the same value as the previous variance. On the other hand we can find Skewness and Kurtosis by calculating the moments of degrees higher than the second

4. Order Statistics

In statistics, the k^{th} order statistic of a statistical sample is equal to its k^{th} smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size n, the n^{th} order statistic (or largest order statistic) is the maximum, that is,

$$X_{(n)}=\max(X_1,X_2,\ldots,X_n).$$

The sample range is the difference between the maximum and minimum. It is clearly a function of the order statistics:

$$Range(X_1,X_2,\ldots,X_n)=X_{(n)}-X_{(1)}.$$

We know that if $X_{(1)},X_{(2)},\ldots,X_{(n)}$ denotes the order statistics of a random sample $X_1,X_2,\ldots,X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$. Then the pdf of $X_{(k)}$ is given by

$$F_{X_{(k)}}(x)=\frac{n!}{(k-1)!(n-k)!}f_X(x)\left[F_X(x)\right]^{k-1}\left[1-F_X(x)\right]^{n-k}$$

$$=\frac{n!}{(k-1)!(n-k)!}f(x)\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^{n-k-i}\left[f_X(x)\right]^{n-k-i}$$

for $k=1,2,\ldots,n$. The pdf of the k^{th} order statistic for transmuted Laplace distribution is given by

a) At $x<0$

$$F_{X_{(k)}}(x)=\frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{2\beta}\ Exp\left(\frac{(\alpha+1)x}{\beta}\right)\right)$$
$$\cdot\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^{n-k-i}\left(\frac{1}{2(\alpha+1)}\right)^{n-k-i}Exp\left(\frac{(\alpha+1)(n-k-i)x}{\beta}\right) \quad (10)$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$F_{X_{(n)}}(x)=\frac{n}{2^n\beta(\alpha+1)^{n-1}}Exp\left(\frac{n(\alpha+1)x}{\beta}\right)$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$F_{X_{(1)}}(x)=\frac{1}{2\beta}\ Exp\left(\frac{(\alpha+1)x}{\beta}\right)$$
$$\times\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{n-1-k}\left(\frac{1}{2(\alpha+1)}\right)^{n-i-1}Exp\left(\frac{(\alpha+1)(n-i-1)x}{\beta}\right).$$

b) At $x>0$

$$F_{X_{(k)}}(x)$$
$$=\frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{2\beta}\left\{Exp\left(\frac{-x}{\beta}\right)\right\}\left\{2-Exp\left(\frac{-\alpha x}{\beta}\right)\right\}\right) \quad (11)$$
$$\times\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^{n-k-i}\left\{\begin{array}{l}Exp\left(\frac{-x}{\beta}\right)\\ -\frac{1}{2(\alpha+1)}Exp\left(\frac{-(\alpha+1)x}{\beta}\right)\end{array}\right\}^{n-k-i}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$F_{X_{(n)}}(x)=\frac{n}{2\beta}\left\{Exp\left(\frac{-x}{\beta}\right)\right\}\left\{2-Exp\left(\frac{-\alpha x}{\beta}\right)\right\}$$
$$\times\left[Exp\left(\frac{-x}{\beta}\right)-\frac{1}{2(\alpha+1)}Exp\left(\frac{-(\alpha+1)x}{\beta}\right)\right]^{n-1}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$F_{X_{(1)}}(x)=\left(\frac{\text{n}}{2\beta}\left\{Exp\left(\frac{-x}{\beta}\right)\right\}\left\{2-Exp\left(\frac{-\alpha x}{\beta}\right)\right\}\right)$$
$$\times\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^{n-i-1}\left\{\begin{matrix}Exp\left(\frac{-x}{\beta}\right)\\ -\frac{1}{2(\alpha+1)}Exp\left(\frac{-(\alpha+1)x}{\beta}\right)\end{matrix}\right\}^{n-i-1}.$$

5. Maximum Likelihood Estimators

In this section we discuss the maximum likelihood estimators (MLE's) and inference for the WLD (β,λ) distribution. Let $x_1,\ldots,x_n$ be a random sample of size n from WLD (β,α) then the likelihood function can be written as

$$L(\theta)=\prod_{i=1}^{n_1}\left[\frac{1}{2\beta}\;Exp\left(\frac{(\alpha+1)x}{\beta}\right)\right]$$
$$\times\prod_{i=1}^{n_2}\left[\frac{1}{2\beta}\left\{Exp\left(\frac{-x}{\beta}\right)\right\}\left\{2-Exp\left(\frac{-\alpha x}{\beta}\right)\right\}\right] \quad (12)$$

Where n_1 is number of the negative observations and n_2 is number of the positive observations.

By accumulation taking logarithm of equation (12), and the log- likelihood function $l(\theta)$ can be written as

$$l(\theta)=-nln(2\beta)-\frac{1}{\beta}\sum_{i=1}^{n}|x_i|+\frac{\alpha}{\beta}\sum_{i=1}^{n_1}x_i+\sum_{i=1}^{n_2}\ln(2-e^{-\frac{\alpha x_i}{\beta}}) \quad (13)$$

Differentiating equation (13) with respect to β and α then equating it to zero. The normal equations become

$$\frac{\partial l(\theta)}{\partial\beta}=\frac{-n}{\beta}+\frac{1}{\beta^2}\sum_{i=1}^{n}|x_i|-\frac{\alpha}{\beta^2}\sum_{i=1}^{n_1}x_i-\frac{\alpha}{\beta^2}\sum_{i=1}^{n_2}\left(\frac{x_ie^{\frac{-\alpha x_i}{\beta}}}{2-e^{-\frac{\alpha x_i}{\beta}}}\right)=0 \quad (14)$$

$$\frac{\partial l(\theta)}{\partial\alpha}=\frac{1}{\beta}\sum_{i=1}^{n_1}x_i+\sum_{i=1}^{n_2}\left(\frac{\frac{x_i}{\beta}e^{\frac{-\alpha x_i}{\beta}}}{2-e^{-\frac{\alpha x_i}{\beta}}}\right)=0。 \quad (15)$$

From (15) we obtain

$$\sum_{i=1}^{n_2}\left(\frac{x_ie^{\frac{-\alpha x_i}{\beta}}}{2-e^{-\frac{\alpha x_i}{\beta}}}\right)=-\sum_{i=1}^{n_1}x_i \quad (16)$$

By putting (16) in (14)

$$\hat{\beta}=\frac{\sum_{i=1}^{n}|x_i|}{n}. \quad (17)$$

And from (17) in (16)

$$\sum_{i=1}^{n_2}\left(\frac{x_ie^{\frac{-\alpha x_i}{\hat{\beta}}}}{2-e^{-\frac{\alpha x_i}{\hat{\beta}}}}\right)=-\sum_{i=1}^{n_1}x_i. \quad (18)$$

Finally by solving (18) numerically we obtain $\hat{\alpha}$.

These solutions will yield the ML estimator for $\hat{\beta}$ and $\hat{\alpha}$ for the two parameters weighted Laplace distribution WLD (β,λ) pdf, all the second order derivatives exist.

Under certain regularity conditions, $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} N(0,I^{-1}(\theta))$ (here $\xrightarrow{d}$ stands for convergence in distribution), where $I(\theta)$ denotes the information matrix given by

$$I(\theta)=E\left(\frac{\partial^2 l(\theta)}{\partial\theta\grave{\partial}\theta}\right).$$

This information matrix $I(\theta)$ may be approximated by the observed information matrix

$$I(\hat{\theta})=\left.\frac{\partial^2 l(\theta)}{\partial\theta\grave{\partial}\theta}\right|_{\theta=\hat{\theta}}.$$

Then, using the approximation

$$\sqrt{n}(\hat{\theta}-\theta)\sim N\left(0,I^{-1}(\hat{\theta})\right)$$

one can carry out tests and find confidence regions for functions of some or all parameters in θ.

Approximate two sided 100(1 $-\gamma$) % confidence intervals for β and α are, respectively, given by

$$\hat{\beta}\pm z_{\gamma/2}\sqrt{I_{11}^{-1}(\theta)}.$$

And

$$\hat{\alpha}\pm z_{\gamma/2}\sqrt{I_{22}^{-1}(\theta)}$$

where z_γ is the upper γ^{th} quantile of the standard normal distribution. Using R, we can easily compute the Hessian matrix and its inverse and hence the standard errors and asymptotic confidence intervals.

We can compute the maximized unrestricted and restricted log-likelihood functions to construct the likelihood ratio (LR) test statistic for testing on some transmuted LD sub-models. For example, we can use the LR test statistic to check whether the WLD distribution for a given data set is statistically superior to the LD distribution. In any case, hypothesis tests of the type $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ can be performed using a LR test. In this case, the LR test statistic for testing H_0 versus H_1 is $\omega=2(\ell(\hat{\theta};x)-\ell(\hat{\theta}_0;x))$, where $\hat{\theta}$ and $\hat{\theta}_0$ are the MLEs under H_1 and H_0 , respectively. The statistic ω is asymptotically (as n $\to\infty$)

distributed as χ_k^2 where k is the length of the parameter vector θ of interest. The LR test rejects H_0 if $\omega > \chi_{k,\alpha}^2$ where $\chi_{k,\alpha}^2$ denotes the upper 100 α% quantile of the χ_k^2 distribution.

6. Application

In this section, we use a real data set to show that the WLD distribution can be a better model than one based on the LD distribution. The data set given in Table 1 taken from Lawless [11] page 228. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life tests and they are:

Table 1. The number of million revolutions before failure for each of the 23 ball bearings in the life tests

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.80, 51.84, 51.96, 54.12, 55.56, 67.80, 68.44, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

We will use this data minus the general average of the experiment; it was 68, in order to fit the data with both Laplace LD and Weighted Laplace WLD.

Table 2. Estimated parameters of the Laplace and Weighted Laplace distribution for the data

Model	Parameter Estimate	Standard Error	$-l(.;x)$
Laplace	$\hat{\beta} = 28.304$	0.278	115.83
Weighted Laplace	$\hat{\beta} = 28.304$ $\hat{\alpha} = 0.515$	0.26 0.078	233.92

The variance covariance matrix of the MLEs under the Weighted Laplace distribution is computed as

$$I^{-1}\left(\hat{\theta}\right)=\begin{bmatrix} 0.11 & -0.011 \\ -0.011 & 0.009 \end{bmatrix}.$$

Thus, the variances of the MLE of β and α are $var(\hat{\beta})$ = 0.41 and $var(\hat{\alpha})$ = 0.0132. Therefore, 95% confidence intervals for β and α are [27.449 - 28.749], and [0.329 - 0.701] respectively.

The LR test statistic to test the hypotheses $H_0: \alpha = 0$ versus $H_1: \alpha \neq 0$ is $\omega = 236.18 > 3.841 = \chi_{1,0.05}^2$, so we reject the null hypothesis.

7. Conclusion

Here we propose a new model, the so-called the Weighted Laplace distribution WDL, which extends the Laplace distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. We derive expansions for moments and for the moment generating function. The estimation of parameters is approached by the method of maximum likelihood; also the information matrix is derived. An application of WDL distribution to real data shows that the new distribution can be used quite effectively to provide better fits than LD distribution.

Also 95% confidence intervals are calculated for the distribution, depending on the standard error for each estimator obtained from the variance covariance matrix, which is the inverse of Fisher matrix.

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