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A Bivariate Distribution with a Two-parameters Exponential Conditional

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Abstract In this paper, a bivariate distribution with a two-parameter exponential conditional is obtained. A multivariate form of the result is also attained under the joint independence of components assumption. A maximum Likelihood method of estimation is provided as well as the intervals of confidence for the parameters of this bivariate distribution. The *pdf* of the order statistics and concommitants are also derived.

Keywords: *a two-parameter exponential distribution, bivariate probability distribution, conditional distribution, concommitants records*

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1. Introduction

The univariate exponential distribution which is analytically very simple plays an important role in describing the life time of a single component [see, e.g., Balakrishnan and Basu (1995)] [1]. The reliability is the domain in which most of the bivariate distributions with exponential marginals arise. Several versions of this bivariate exponential distribution are encountered in the literature and have been used for modeling the two components systems. Indeed, a complete class of bivariate distribution respectively with normal and exponential conditional were identified, Castillo and Galombos (1987a) [4], Barry C. Arnold and David Strauss (1988) [3].

The marginal densities of the bivariate exponential may not be exponential. It can be a mixture of exponential. In such case the bivariate distribution is often called a bivariate exponential mixture distribution (see, Kotz et al. [8]). Many authors proposed the multivariate form of the exponential distribution (see, Johnson et al. [7]).

Recently Filus and Filus [6] have proposed for modeling lifetimes of multi-component system, a new class of probability distributions based upon a linear combination of independent random variables.

In this paper, we define a bivariate distribution with a two-parameters (a, b) exponential conditional which can be used for modeling lifetime of two component system.

The bivariate distribution with conditional a two-parameters exponential distribution is introduced in section 2 below with some characteristics such as the marginal densities, the moments, the product moments, the conditional moments, the moment generating function, the survivor distribution and the entropies. In section 3, we infer about the parameters of our bivariate distribution by giving their maximum likelihood likelihood estimators (*MLEs*) and intervals of confidence.

In section 4, we introduce the distribution of the concomitants of the order statistic. Finally in section 5 the multivariate case is studied with its related properties.

2. The Bivariate Distribution with Conditional a Two-parameters Exponential Distribution

Let X be a two-parameter exponential distribution random variable. The probability density function ($p.d.f$) of X is given by

$$f_X(x \mid a,b) = \frac{1}{b} e^{-\frac{x-a}{b}}, \quad x \geq a, \quad b > 0 \tag{2.1}$$

with, $E(X) = a + b$ and $V(X) = b^2$.

The cumulative distribution function of X is given by

$$F_X(x \mid a,b) = 1 - e^{-\frac{x-a}{b}}, \quad x \geq a, \quad b > 0. \tag{2.2}$$

Now, let Y be a random variable such that the distribution of Y given X is a two-parameters (x,c) exponential distribution. The $p.d.f$ of $Y \mid X$ is given by

$$f_{Y|X}(y \mid x,c) = \frac{1}{c} e^{-\frac{y-x}{c}}, \quad y \geq x, \quad c > 0 \quad (c \neq b). \tag{2.3}$$

Thus the joint density of the random variables X and Y defined above is given by

$$f_{X,Y}(x,y) = f_{Y|X}(y|x,c)\, f_X(x|a,b) = \frac{e^{\frac{a}{b}}}{bc} e^{-(\frac{1}{b}-\frac{1}{c})x} e^{-\frac{y}{c}}, a \le x \le y, \quad b,c > 0. \tag{2.4}$$

It can be easily verified that equation (2.4) integrates to 1, so it is a joint probability distribution.

The plot of this joint distribution for different values of a, b, and c is given in Figure 1.

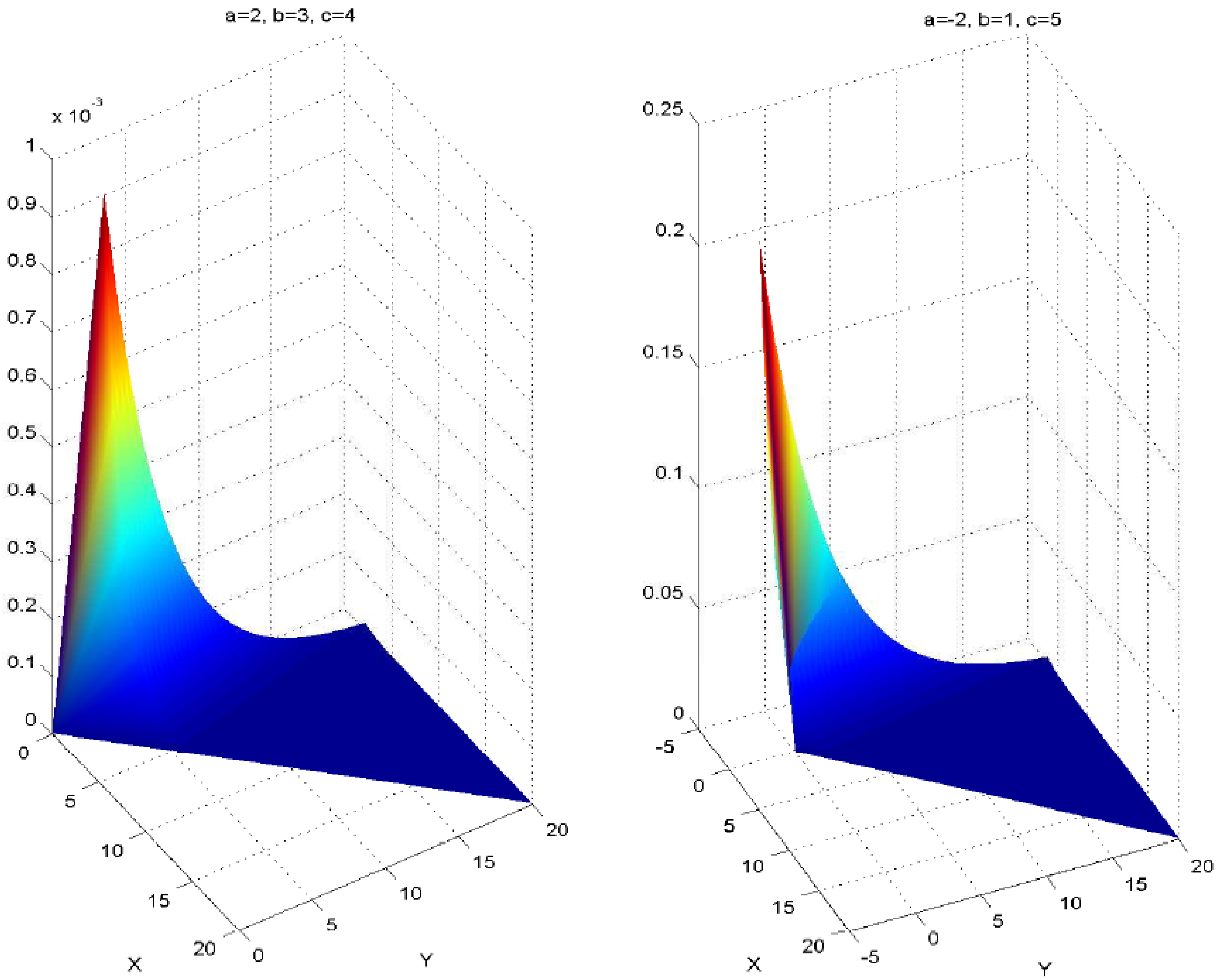

Figure 1. Graphs of $f_{X,Y}(x,y)$

Thus the cumulative distribution of the random variables X and Y is

$$F_{X,Y}(x,y) = \frac{ce^{\frac{a}{b}}}{c-b}\left[e^{-(\frac{1}{b}-\frac{1}{c})x} - e^{-a(\frac{1}{b}-\frac{1}{c})}\right]\left[e^{-\frac{y}{c}} - e^{-\frac{a}{c}}\right],$$

$a \le x \le y, b,c > 0.$

2.1. Marginal and Moments of *Y*

As the marginal of X is given by (2.1), the marginal of Y is derived as follows

Theorem 2.1.

$$f_Y(y) = \int_a^y f_{X,Y}(x,y)\,\mathrm{d}x = \frac{1}{c-b}\left[e^{-\frac{(y-a)}{c}} - e^{-\frac{(y-a)}{b}}\right],$$

$a \le y, \quad b,c > 0.$

Proof.

$$\begin{aligned} f_Y(y) &= \int_a^y f_{X,Y}(x,y)\,\mathrm{d}x \\ &= \frac{e^{\frac{a}{b}}}{bc} e^{-\frac{y}{c}} \int_a^y e^{-x(\frac{1}{b}-\frac{1}{c})}\,\mathrm{d}x \\ &= \frac{e^{\frac{a}{b}}}{bc} e^{-\frac{y}{c}} \left[-\frac{1}{\frac{1}{b}-\frac{1}{c}} e^{-x(\frac{1}{b}-\frac{1}{c})}\Big|_a^y\right] \\ &= \frac{1}{c-b}\left[e^{-\frac{(y-a)}{c}} - e^{-\frac{(y-a)}{b}}\right]. \end{aligned} \tag{2.5}$$

Consequently the cumulative distribution of Y is

$$F_Y(y) = 1 - \frac{1}{c-b}\left(ce^{-\frac{y-a}{c}} - be^{-\frac{y-a}{b}}\right), \quad a \le y, \quad b,c > 0.$$

Remark 2.2. *The marginal* $f_Y(y)$ *of* $f_{X,Y}(x,y)$ *is not an* $Exp(\alpha,\beta)$ *but a mixture of exponential, so* $f_{X,Y}(x,y)$ *is a bivariate exponential mixture distribution.*

The m^{th} moments of Y are given by

Theorem 2.3. *The* m^{th} *moments of* Y *are:*

$$E(Y^m)=\frac{1}{c-b}\sum_{k=0}^{m}k!\binom{m}{k}a^{m-k}(c^{k+1}-b^{k+1}).$$

Proof.

$$E(Y^m)=\frac{1}{c-b}\int_0^{\infty}y^m\left[e^{-\frac{(y-a)}{c}}-e^{-\frac{(y-a)}{b}}\right]\mathrm{d}y$$

$$\int_a^{\infty}y^m e^{-\frac{(y-a)}{c}}\mathrm{d}y=\int_0^{\infty}(u+a)^m e^{-\frac{u}{c}}\mathrm{d}u,\ (\text{with } u=y-a)$$

$$=\int_0^{\infty}\sum_{k=0}^{m}\binom{m}{k}u^k a^{m-k}e^{-\frac{u}{c}}\mathrm{d}u=\sum_{k=0}^{m}\binom{m}{k}a^{m-k}\int_0^{\infty}u^k e^{-\frac{u}{c}}\mathrm{d}u$$

$$=\sum_{k=0}^{m}\binom{m}{k}a^{m-k}c^{k+1}\int_0^{\infty}t^k e^{-t}\mathrm{d}t,\ (\text{with } t=\frac{u}{c})$$

$$=\sum_{k=0}^{m}\binom{m}{k}a^{m-k}c^{k+1}\Gamma(k+1)=\sum_{k=0}^{m}k!\binom{m}{k}a^{m-k}c^{k+1}$$

By analogy

$$\int_a^{\infty}y^m e^{-\frac{(y-a)}{b}}\mathrm{d}y=\sum_{k=0}^{m}k!\binom{m}{k}a^{m-k}b^{k+1},\ \text{then:}$$

$$E(Y^m)=\frac{1}{c-b}\sum_{k=0}^{m}k!\binom{m}{k}a^{m-k}(c^{k+1}-b^{k+1}).\qquad(2.6)$$

Remark 2.4. *From* (2.6) *we deduce that:*

1. $E(Y)=a+b+c.$
2. $V(Y)=b^2+c^2.$

The (*p, q*)*th* joint moment of (*X, Y*) can also be obtained as follows

Theorem 2.5.

$$E(X^pY^q)=\int_a^{\infty}\int_a^{y}x^p y^q f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$

$$=\frac{e^{\frac{a}{b}}}{bc}\int_a^{\infty}\left[\int_a^{y}x^p e^{-(\frac{1}{b}-\frac{1}{c})x}\mathrm{d}x\right]y^q e^{-\frac{y}{c}}\mathrm{d}y$$

$$=e^{a(\frac{1}{b}-\frac{1}{c})}\sum_{n=0}^{\infty}\sum_{k=0}^{q}\left\{\left[\binom{n+p+q+1}{k}-\binom{q}{k}\right]\times\frac{k!(b-c)^n a^{n+p+q-k+1}}{n!(n+p+1)b^{n+1}c^{n-k}}\right\}\qquad(2.7)$$

$$+e^{a(\frac{1}{b}-\frac{1}{c})}\sum_{n=0}^{\infty}\sum_{k=q+1}^{n+p+q+1}\left\{\binom{n+p+q+1}{k}\times\frac{k!(b-c)^n a^{n+p+q-k+1}}{n!(n+p+1)b^{n+1}c^{n-k}}\right\}$$

Proof.

$$E(X^pY^q)=\frac{e^{\frac{a}{b}}}{bc}\int_a^{\infty}\left[\int_a^{y}x^p e^{-(\frac{1}{b}-\frac{1}{c})x}\mathrm{d}x\right]y^q e^{-\frac{y}{c}}\mathrm{d}y.$$

Expanding $e^{-(\frac{1}{b}-\frac{1}{c})x}$ in power series and putting $\alpha=\frac{1}{b}-\frac{1}{c}$, and $\beta=\frac{1}{c}$ we get

$$E(X^pY^q)$$

$$=\frac{e^{\frac{a}{b}}}{bc}\sum_{n=0}^{\infty}\frac{(-1)^n\alpha^n}{n!(n+p+1)}\int_a^{\infty}y^q\left(y^{n+p+1}-a^{n+p+1}\right)e^{-\beta y}\mathrm{d}y.$$

Let $I_1=\int_a^{\infty}y^{n+p+q+1}e^{-\beta y}\mathrm{d}y$ and

$$I_2=a^{n+p+1}\int_a^{\infty}y^q e^{-\beta y}\mathrm{d}y$$

$$I_1=e^{-a\beta}\int_a^{\infty}y^{n+p+q+1}e^{-\beta(y-a)}\mathrm{d}y$$

$$=\frac{e^{-\beta a}}{\beta^{n+p+q+2}}\int_0^{\infty}(u+\beta a)^{n+p+q+1}e^{-u}\mathrm{d}u$$

$$\left(\text{with } u=\beta(y-a)\right)$$

$$=\frac{e^{-\beta a}}{\beta^{n+p+q+2}}\int_0^{\infty}\sum_{k=0}^{n+p+q+1}\left[\binom{n+p+q+1}{k}\times u^k(\beta a)^{n+p+q-k+1}e^{-u}\right]\mathrm{d}u$$

$$=\frac{e^{-\beta a}}{\beta^{n+p+q+2}}\sum_{k=0}^{n+p+q+1}\left[\binom{n+p+q+1}{k}\times(\beta a)^{n+p+q-k+1}\right]\int_0^{\infty}u^k e^{-u}\mathrm{d}u$$

$$=\frac{e^{-\beta a}a^{n+p+q+1}}{\beta}\sum_{k=0}^{n+p+q+1}\binom{n+p+q+1}{k}\frac{k!}{(a\beta)^k}$$

$$=A\sum_{k=0}^{n+p+q+1}\binom{n+p+q+1}{k}\frac{k!}{(a\beta)^k}$$

$$\left(\text{with } A=\frac{e^{-\beta a}a^{n+p+q+1}}{\beta}\right).\qquad(2.8)$$

By the same way we prove that $I_2=A\sum_{k=0}^{q}\binom{q}{k}\frac{k!}{(a\beta)^k}$, then

$$E(X^pY^q)=\frac{e^{\frac{a}{b}}}{bc}\sum_{n=0}^{n=\infty}\frac{(-1)^n\alpha^n}{n!(n+p+1)}\left(I_1-I_2\right)$$

$$=\frac{e^{\frac{a}{b}}}{bc}\sum_{n=0}^{n=\infty}\frac{(-1)^n\alpha^n}{n!(n+p+1)}A\left[\sum_{k=0}^{n+p+q+1}\binom{n+p+q+1}{k}\frac{k!}{(a\beta)^k}-\sum_{k=0}^{q}\binom{q}{k}\frac{k!}{(a\beta)^k}\right]$$

$$= \frac{e^{\frac{a}{b}}}{bc} \sum_{n=0}^{n=\infty} \frac{(-1)^n A\alpha^n}{n!(n+p+1)}$$

$$\times \left[\begin{array}{l} \sum_{k=0}^{q} \left[\binom{n+p+q+1}{k} - \binom{q}{k} \right] \frac{k!}{(a\beta)^k} \\ + \sum_{k=q+1}^{n+p+q+1} \binom{n+p+q+1}{k} \frac{k!}{(a\beta)^k} \end{array} \right]$$

$$= e^{a(\frac{1}{b}-\frac{1}{c})} \sum_{n=0}^{\infty} \sum_{k=0}^{q} \left\{ \begin{array}{l} \left[\binom{n+p+q+1}{k} - \binom{q}{k} \right] \\ \times \frac{k!(b-c)^n a^{n+p+q-k+1}}{n!(n+p+1)b^{n+1}c^{n-k}} \end{array} \right\} \quad (2.9)$$

$$+ e^{a(\frac{1}{b}-\frac{1}{c})} \sum_{n=0}^{\infty} \sum_{k=q+1}^{n+p+q+1} \left\{ \begin{array}{l} \binom{n+p+q+1}{k} \\ \times \frac{k!(b-c)^n a^{n+p+q-k+1}}{n!(n+p+1)b^{n+1}c^{n-k}} \end{array} \right\}$$

2.2. The Moment Generating Function

The moment generating function of (X, Y) is given as

$$M(t_1,t_2) = E\left\{e^{(t_1X+t_2Y)}\right\}$$

$$= \int_a^{\infty} e^{t_1x+t_2y} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y$$

$$= \int_a^{\infty}\int_a^{y} e^{(t_1x+t_2y)} \frac{e^{\frac{a}{b}}}{bc} e^{-(\frac{1}{b}-\frac{1}{c})x} e^{-\frac{y}{c}} \mathrm{d}x\mathrm{d}y \quad (2.10)$$

$$= \frac{e^{a(t_1+t_2)}}{(1-bt_1-bt_2)(1-ct_2)}$$

The product moment exists if $t_1+t_2 < \frac{1}{b}$ with $t_2 < \frac{1}{c}$.

From (2.10) we can deduce:

1.

$$E(XY) = \left[\frac{\partial}{\partial t_1} \left(\frac{\partial M(t_1,t_2)}{\partial t_2} \right) \right] \Big|_{t_1=t_2=0}$$

$$= (a+b)^2 + b^2 + c(a+b)$$

2. $cov(X,Y) = E(XY) - E(X)E(Y) = b^2$

3. $\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{b^2}{\sqrt{b^2+c^2}\sqrt{b^2}} = \frac{b}{\sqrt{b^2+c^2}}$,

as $\rho(X,Y) > 0$, X and Y are positively correlated.

4. The matrix of Variance-Covariance of X and Y is

$$Cov(X,Y) = \begin{pmatrix} b^2 & b^2 \\ b^2 & b^2+c^2 \end{pmatrix}.$$

2.3. Conditional Moments

The conditional distribution of $X \mid Y$ ant that of $Y \mid X$ are

$$f_{X|y}(x \mid y) = \frac{f(x,y)}{f(y)} = \frac{c-b}{bc} e^{-a(\frac{1}{c}-\frac{1}{b})} e^{-(\frac{1}{c}-\frac{1}{b})(y-x)}, \quad (2.11)$$

$y \ge x,\ b,c > 0$

and

$$f_{Y|x}(y \mid x) = \frac{1}{c} e^{-\frac{y-x}{c}},\ y \ge x,\ c > 0. \quad (2.12)$$

Using (2.11) we get the *pth* conditional moments of X as

Theorem 2.6.

$$E(X^p \mid y) = \int_a^y x^p f(x \mid y)\mathrm{d}x$$

$$= e^{-a(\frac{1}{c}-\frac{1}{b})} e^{-(\frac{1}{c}-\frac{1}{b})y} \times \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{c-b}{bc} \right)^{n+1} \frac{y^{n+p+1} - a^{n+p+1}}{n+p+1}.$$

Proof.

$$E(X^p \mid y) = \int_a^y x^p f(x \mid y)\mathrm{d}x$$

$$= \frac{c-b}{bc} e^{-a(\frac{1}{c}-\frac{1}{b})} e^{-(\frac{1}{c}-\frac{1}{b})y} \int_a^y x^p e^{(\frac{1}{c}-\frac{1}{b})x} \mathrm{d}x$$

$$= K \int_a^y x^p \sum_{n=0}^{\infty} \frac{\left((\frac{1}{c}-\frac{1}{b})x \right)^n}{n!} \mathrm{d}x$$

$$\left(\text{with } K = \frac{c-b}{bc} e^{-a(\frac{1}{c}-\frac{1}{b})} e^{-(\frac{1}{c}-\frac{1}{b})y} \right) \quad (2.13)$$

$$= K \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{b-c}{bc} \right)^n \left[\int_a^y x^{n+p} \mathrm{d}x \right]$$

$$= e^{-a(\frac{1}{c}-\frac{1}{b})} e^{-(\frac{1}{c}-\frac{1}{b})y} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{c-b}{bc} \right)^{n+1} \frac{\begin{pmatrix} y^{n+p+1} \\ -a^{n+p+1} \end{pmatrix}}{n+p+1}.$$

Similarly, using (2.12) we get the *qth* conditional moments of Y as

Theorem 2.7.

$$E(Y^q \mid x) = \int_a^{\infty} y^q f(y \mid x)\mathrm{d}y = e^{\frac{x-a}{c}} \sum_{k=0}^{q} k! \binom{q}{k} c^k a^{q-k}$$

Proof.

$$E(Y^q \mid x) = \int_a^{\infty} y^q f(y \mid x)\mathrm{d}y = \int_0^{\infty} \frac{y^q}{c} e^{-\frac{y-x}{c}} \mathrm{d}y$$

$$= \frac{e^{\frac{x}{c}}}{c} \int_a^{\infty} y^q e^{-\frac{y}{c}} \mathrm{d}y = \frac{e^{\frac{x-a}{c}}}{c} \int_a^{\infty} y^q e^{-\frac{y-a}{c}} \mathrm{d}y$$

$$= e^{\frac{x-a}{c}} \int_0^{\infty} (ct+a)^q e^{-t} \mathrm{d}t \qquad \left(\text{with } \frac{y-a}{c} = t \right) \quad (2.14)$$

$$= e^{\frac{x-a}{c}} \int_0^{\infty} \left[\sum_{k=0}^{q} \binom{q}{k} (ct)^k a^{q-k} \right] e^{-t} \mathrm{d}t$$

$$= e^{\frac{x-a}{c}} \sum_{k=0}^{q} \binom{q}{k} c^k a^{q-k} \int_0^{\infty} t^k e^{-t} \mathrm{d}t = e^{\frac{x-a}{c}} \sum_{k=0}^{q} k! \binom{q}{k} c^k a^{q-k}$$

Remark 2.8. *From* (2.13) *and* (2.14) *we can easily obtain the conditional means and variances of* $X \mid y$ *and* $Y \mid x$.

2.4. The Joint Survivor Function

For the mixture distribution (2.4) the joint survivor function $S(x,y) = \mathbb{P}(X > x, Y > y)$ which can be used in the reliability study of systems, is given by

$$S(x,y) = 1 - F_X(x) - F_Y(y) + F_{X,Y}(x,y)$$

$$= 1 - \left[1 - e^{-\frac{x-a}{b}}\right] - \left[1 - \frac{1}{c-b}\left(ce^{-\frac{y-a}{c}} - be^{-\frac{y-a}{b}}\right)\right]$$

$$+ \frac{ce^{\frac{a}{b}}}{c-b}\left[e^{-(\frac{1}{b}-\frac{1}{c})x} - e^{-a(\frac{1}{b}-\frac{1}{c})}\right]\left[e^{-\frac{y}{c}} - e^{-\frac{a}{c}}\right] \quad (2.15)$$

$$= \frac{ce^{\frac{a}{b}}}{c-b}\left[e^{-(\frac{1}{b}-\frac{1}{c})x} - e^{-a(\frac{1}{b}-\frac{1}{c})}\right]\left[e^{-\frac{y}{c}} - e^{-\frac{a}{c}}\right]$$

$$+ e^{-\frac{x-a}{b}} + \frac{1}{c-b}\left(ce^{-\frac{y-a}{c}} - be^{-\frac{y-a}{b}}\right) - 1.$$

The failure rates of the random variables X and Y having *p.d.f* $f_X(x)$ and $f_Y(y)$ given by (2.1) and (2.5), respectively, are

$$r_X(t) = \frac{f_X(t)}{1 - F_X(t)} = \frac{1}{b}$$

and

$$r_Y(t) = \frac{f_Y(t)}{1 - F_Y(t)} = \frac{e^{-\frac{(t-a)}{c}} - e^{-\frac{(t-a)}{b}}}{ce^{-\frac{(t-a)}{c}} - be^{-\frac{(t-a)}{b}}}.$$

The plot of the failure rate of Y for different values of a, b, and c is given in Figure 2.

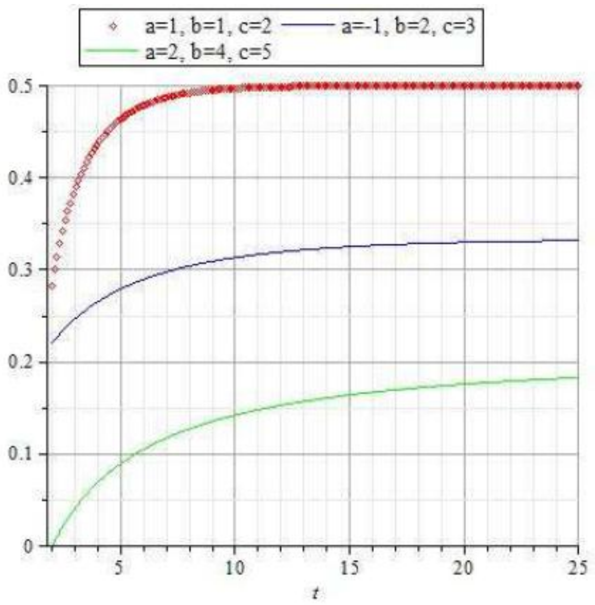

Figure 2. Graphs of $r_Y(t)$

2.5. Entropies

In this section we introduce the entropy between X and Y which is defined as $H(x,y) = E[-ln\{f(X,Y)\}]$ and interpreted as the quantity of information on X we gain by learning Y. So, for the bivariate mixture distribution the entropy is

$$H(x,y) = E[-ln\{f_{X,Y}(X,Y)\}]$$

$$= -E\left[ln\left(\frac{e^{\frac{a}{b}}}{bc}e^{-(\frac{1}{b}-\frac{1}{c})X}e^{-\frac{Y}{c}}\right)\right]$$

$$= ln(bc) - \frac{a}{b} + \left(\frac{1}{b} - \frac{1}{c}\right)E(X) + \frac{1}{c}E(Y) \quad (2.16)$$

$$= ln(bc) - \frac{a}{b} + \left(\frac{1}{b} - \frac{1}{c}\right)(a+b) + \frac{1}{c}(a+b+c)$$

$$= ln(bc) + 2.$$

3. Inference

3.1. Parameters Estimation For the Bivariate Distribution with *Exp*(*a*,*b*) Conditional

We introduce here, the maximum Likelihood estimation for the bivariate model.

Let (x_i, y_i) for $i = 1,...,n$ be a sample of size n from the bivariate distribution defined in (2.4). Then the log likelihood function is

$$l(a,b,c)$$

$$= \frac{na}{b} - n\ln(b) - n\ln(c) - \frac{1}{b}\sum_{i=1}^{n} x_i - \frac{1}{c}\sum_{i=1}^{n}(y_i - x_i) \quad (3.17)$$

$$= \frac{na}{b} - n\ln(b) - n\ln(c) - \frac{n\bar{x}}{b} - \frac{n(\bar{y} - \bar{x})}{c}.$$

We have to maximize this function under the constraints $a \le x_i \le y_i$ for $i = 1,...,n$ (5.14), $b > 0$, and $c > 0$.

Theorem 3.1. *The maximum likelihood estimators of* a, b, *and* c *are given by*

$$\hat{a} = X_{(1)} = \min_{1 \le i \le n} X_i,\ \hat{b} = \bar{X} - X_{(1)},\ \text{and } \hat{c} = \bar{Y} - \bar{X}.$$

Proof. From (2:4) we deduce that $x_{(1)} \le \bar{x} \le \bar{y}$.

More, it will be assumed that

1. $\exists i, j \le n$ such that $x_i \ne x_j$ not all x_i equal

2. $\exists k \le n$ such that $y_k \ne x_k$ (which means $y_k > x_k$).

So $x_{(1)} < \bar{x} < \bar{y}$, and the unique constraint on a is $a \le x_i$ \,for all $1 \le i \le n$, which can be written as $a \le x_{(1)}$.

The function $l(a,b,c)$ is increasing linear with respect to the variable a when we fixe b>0 and c>0. Therefore its maximum is attained for $a = x_{(1)}$. So we have just to maximize the following function with respect to the variables b and c

$$g(b,c)=\frac{1}{n}l(x_{(1)},b,c)$$
$$=\frac{x_{(1)}}{b}-\ln(b)-\ln(c)-\frac{\overline{x}}{b}-\frac{\overline{y}-\overline{x}}{c} \qquad (3.18)$$
$$=-\frac{\overline{x}-x_{(1)}}{b}-\ln(b)-\ln(c)-\frac{\overline{y}-\overline{x}}{c}.$$

This function g can be written as $g(b,c)=-g_1(b)-g_2(c)$ with $g_1(b)=\frac{\overline{x}-x_{(1)}}{b}+\ln(b)$, $g_2(c)=\frac{\overline{y}-\overline{x}}{c}+\ln(c)$.

Maximize g with respect to (b,c) is equivalent to minimize g_1 with respect to b and minimize g_2 with respect to c.

Those two functions g_1 and g_2 are of the form

$$h(x)=\frac{\alpha}{x}+\ln(x),\ \alpha>0.$$

($\alpha=\overline{x}-x_{(1)}$ for g_1, and $\alpha=\overline{x}-\overline{y}$ for g_2).

We can easily prove that h has a unique global minimum on $]0,\infty[$ attained at x_0 such that $h'(x_0)=-\frac{\alpha}{x_0^2}+\frac{1}{x_0}=0$ i.e.: $x_0=\alpha$.

So $b_0=\overline{x}-x_{(1)}>0$ and $c_0=\overline{y}-\overline{x}>0$ are the global minimum for $g_1(b)$ and $g_2(c)$ respectively.

Therefore the function l has a global maximum (under the constraints) attained at

$$(a_0,b_0,c_0)=(x_{(1)},\overline{x}-x_{(1)},\overline{y}-\overline{x}).$$

So $X_{(1)},\overline{X}-X_{(1)}$ and $\overline{Y}-\overline{X}$ are the MLEs of a, b, and c respectively.

Lawless (1982) [6] proved that $\hat{a}=X_{(1)}$ and $\hat{b}=\overline{X}-X_{(1)}$ are independent with

$$\frac{2n(\hat{a}-a)}{b}\sim\chi_2^2 \text{ and } \frac{2n\hat{b}}{b}\sim\chi_{2n-2}^2. \qquad (3.19)$$

Using (3.19) and $\hat{c}=\overline{Y}-\overline{X}$ we get the following results:

1. $E(\hat{a})=\frac{b}{n}+a$. ($\hat{a}$ is a positively biased estimator of a with bias equal $\frac{b}{n}$)

2. $V(\hat{a})=\frac{b^2}{n^2}$ and $MSE(\hat{a})=V(\hat{a})+(bias)^2=\frac{2b^2}{n^2}$.

3. $E(\hat{b})=\frac{(n-1)b}{n}$. ($\hat{b}$ is a negatively biased estimator of b with bias equal $(-\frac{b}{n})$)

4. $V(\hat{b})=\frac{(n-1)b^2}{n^2}$ and

$$MSE(\hat{b})=V(\hat{b})+(bias)^2=\frac{(n-2)b^2}{n^2}.$$

5. $E(\hat{c})=E(\overline{Y}-\overline{X})=E(Y)-E(X)=c$. ($\hat{c}$ is an unbiased estimator of c)

6.

$$V(\hat{c})=MSE(\hat{c})=V(\overline{Y})+V(\overline{X})-2cov(\overline{X},\overline{Y})$$
$$=\frac{V(Y)}{n}+\frac{V(X)}{n}-\frac{2}{n}cov(X,Y)=\frac{c^2}{n}.$$

Remark 3.2.

1. $\lim_{n\to\infty}MSE(\hat{a})=\lim_{n\to\infty}MSE(\hat{b})=\lim_{n\to\infty}MSE(\hat{c})=0$, then *$\hat{a}$, $\hat{b}$ and $\hat{c}$ are consistent estimators of a, b and c respectively.*

2. *$\hat{a}$ and $\hat{b}$ are asymptotically unbiased estimators of a and b respectively.*

3.2. Intervals of Confidence for the Parameters of the Bivariate Distribution with *Exp*(*a*,*b*) Conditional

We introduce here, the intervals of confidence for the three parameters a, b, and c.

We can use the pivotal quantity $\frac{2n\hat{b}}{b}$ in (3.19) to make inference on b, and a $(1-\alpha)$ confidence interval for b is given by

$$\left(\frac{2n\hat{b}}{\chi^2_{2n-2,1-\frac{\alpha}{2}}},\frac{2n\hat{b}}{\chi^2_{2n-2,\frac{\alpha}{2}}}\right).$$

It follows also from (3.19) that

$$\frac{\hat{a}-a}{\hat{b}}\sim\frac{1}{n-1}F_{2,2n-2}.$$

By the same way, using the pivotal quantity $\frac{\hat{a}-a}{\hat{b}}$, a $(1-\alpha)$ confidence interval for a can be derived as

$$\left(\hat{a}-\frac{\hat{b}}{n-1}F_{2,2n-2,1-\frac{\alpha}{2}},\hat{a}-\frac{\hat{b}}{n-1}F_{2,2n-2,\frac{\alpha}{2}}\right).$$

Also for n enough large $(n\geq 30)$, $\hat{c}=\overline{Y}-\overline{X}$ follows $N\left(c,\frac{c^2}{n}\right)$ and then

$$\frac{\hat{c}-c}{\sqrt{\frac{c^2}{n}}}\sim N(0,1).$$

So for $V(\hat{c})=\frac{\hat{c}^2}{n}$ as an estimator of $V(\hat{c})$, a $(1-\alpha)$ confidence interval for c can be derived and it is given by

$$\left(\hat{c}-z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{c}^2}{n}},\hat{c}+z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{c}^2}{n}}\right)=\left(\hat{c}-z_{\frac{\alpha}{2}}\frac{\hat{c}}{\sqrt{n}},\hat{c}+z_{\frac{\alpha}{2}}\frac{\hat{c}}{\sqrt{n}}\right).$$

4. Concomitants of Order Statistics

In this section we introduce the distribution of the concomitants of the order statistic for the bivariate exponential mixture distribution. The density of probability of the *rth* concomitant is given by [5] as

$$g_{[r:n]}(y)=\int_{-\infty}^{\infty} f(y\mid x)\, f_{r:n}(x)\mathrm{d}x$$

where $f_{r:n}(x)$ is the density function of the *rth* order statistic for the variable *X* given by

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!} f(x)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r}.$$

Given (2.1), (2.2), and (2.3), the distribution of the *rth* order statistic for *X* is

$$f_{r:n}(x)$$

$$=\frac{n!}{b(r-1)!(n-r)!}\left[e^{-\frac{x-a}{b}}\right]^{n-r+1}\left[1-e^{-\frac{x-a}{b}}\right]^{r-1}$$

$$=\frac{n!e^{\frac{a(n-r+1)}{b}}}{b(r-1)!(n-r)!}e^{-\frac{x(n-r+1)}{b}}\left[1-e^{-\frac{x-a}{b}}\right]^{r-1} \quad (4.20)$$

$$=Ke^{-\frac{x(n-r+1)}{b}}\left[1-e^{-\frac{x-a}{b}}\right]^{r-1}\left(K=\frac{n!e^{\frac{a(n-r+1)}{b}}}{b(r-1)!(n-r)!}\right)$$

Theorem 4.1. *The density of the rth concomitant is given by*

$$g_{[r:n]}(y)=\int_a^y f(y\mid x)\, f_{r:n}(x)\mathrm{d}x$$

$$=\frac{n!}{(r-1)!(n-r)!}$$

$$\times\sum_{k=0}^{r-1}\frac{(-1)^k\binom{r-1}{k}\left[e^{-\frac{y-a}{c}}-e^{-\frac{y-a}{b}(n-r+k+1)}\right]}{c(n-r+k+1)-b}.$$

Proof.

$$g_{[r:n]}(y)=\int_a^y f(y\mid x)\, f_{r:n}(x)\mathrm{d}x$$

$$=K\int_a^y \frac{1}{c}e^{-\frac{y-x}{c}}e^{-x(\frac{n-r+1}{b})}\left[1-e^{-\frac{x-a}{b}}\right]^{r-1}\mathrm{d}x$$

$$=\frac{K}{c}e^{-\frac{y}{c}}\int_a^y e^{-x(\frac{n-r+1}{b}-\frac{1}{c})}\left[1-e^{\frac{a}{b}}e^{-\frac{x}{b}}\right]^{r-1}\mathrm{d}x$$

$$=\frac{K}{c}e^{-\frac{y}{c}}\sum_{k=0}^{r-1}\binom{r-1}{k}(-1)^k e^{\frac{ka}{b}}\int_a^y e^{-x(\alpha+\frac{k}{b})}\mathrm{d}x$$

$$\left(\alpha=\frac{n-r+1}{b}-\frac{1}{c}\right)$$

$$=\frac{K}{c}e^{-\frac{y}{c}}\sum_{k=0}^{r-1}b\binom{r-1}{k}(-1)^k\frac{e^{\frac{ka}{b}}}{\alpha b+k}\left[e^{-a(\alpha+\frac{k}{b})}-e^{-y(\alpha+\frac{k}{b})}\right]$$

$$=\frac{n!e^{\frac{a(n-r+1)}{b}}}{(r-1)!(n-r)!}e^{-\frac{y}{c}}$$

$$\times\sum_{k=0}^{r-1}\binom{r-1}{k}(-1)^k\frac{e^{\frac{ka}{b}}}{c(n-r+k+1)-b}$$

$$\times\left[e^{-a(\frac{n-r+1}{b}-\frac{1}{c})}-e^{-y(\frac{n-r+1}{b}-\frac{1}{c})}\right]$$

$$=\frac{n!}{(r-1)!(n-r)!}\sum_{k=0}^{r-1}\frac{(-1)^k\binom{r-1}{k}\left[e^{-\frac{y-a}{c}}-e^{-\frac{y-a}{b}(n-r+k+1)}\right]}{c(n-r+k+1)-b}$$

The *pth* moment of the concomitant of the order statistic is given by

Theorem 4.2.

$$Y^p_{[r:n]})=\int_a^\infty y^p g_{[r:n]}(y)\mathrm{d}y$$

$$=\frac{n!}{(r-1)!(n-r)!}\sum_{k=0}^{r-1}\sum_{m=0}^{p}\left\{\frac{m!(-1)^k\binom{r-1}{k}\binom{p}{m}a^{p-m}}{c(n-r+k+1)-b}\times\left[c^{m+1}-\frac{b^{m+1}}{(n-r+m+1)^{m+1}}\right]\right\}.$$

Proof. Using the same techniques of integrations as in theorem 4.1 above we get our result.

Remark 4.3. *From theorem 4.2 we can deduce the expected value and variance of* $Y_{[r:n]}$.

The expression of the survivor function

$$S_{Y_{[r:n]}}(t)=\mathbb{P}\left(Y_{[r:n]}>t\right)$$

of $Y_{[r:n]}$ is.

Theorem 4.4.

$$E(S_{Y_{[r:n]}}(t)=\frac{n!}{(r-1)!(n-r)!}\sum_{k=0}^{r-1}\frac{(-1)^k\binom{r-1}{k}}{c(n-r+k+1)-b}\times\int_t^\infty\left(e^{-\frac{y-a}{c}}-e^{-\frac{y-a}{b}(n-r+k+1)}\right)\mathrm{d}y$$

$$=\frac{n!}{(r-1)!(n-r)!}\sum_{k=0}^{r-1}\frac{(-1)^k\binom{r-1}{k}}{c(n-r+k+1)-b}$$

$$\times\left[ce^{-\frac{t-a}{c}}-\frac{b}{n-r+k+1}e^{-\frac{t-a}{b}(n-r+k+1)}\right].$$

Proof. Obvious.

5. Multivariate Case

Let $X, Y_1, ..., Y_n$ be $n+1$ random variables, the multivariate case is built as

$$\begin{aligned} &X : Exp(a,b) \\ &Y_1 \mid X : Exp(\alpha_1 x, \beta_1) \\ &Y_2 \mid X : Exp(\alpha_2 x, \beta_2) \\ &\vdots \\ &Y_n \mid X : Exp(\alpha_n x, \beta_n), \end{aligned}$$

where $Y_i \mid X$ and $Y_j \mid X$ are independent random variables for $i \neq j$ and $i, j = 1, 2, ..., n$ and $\alpha_i \neq 0$, $i = 1, .., n$. HhUsing the same arguments as in the univariate case above, the joint component model is built and the marginal density function for each random variable Y_i is derived. In general, Y_i has the following density

$$f_Y(y_i) = \frac{1}{\beta_i - b\alpha_i}\left(e^{-\frac{y_i - a\alpha_i}{\beta_i}} - e^{-\frac{y_i - a\alpha_i}{b}} \right), \quad (5.21)$$

$i = 1, 2, ..., n$ with $a\alpha_i \le y_i, b > 0, \beta_i > 0$.

Based on the independence assumption of the above model, the joint density of $Y_1, ..., Y_n$ has the following form

$$\begin{aligned} &f_{X,Y_1,...,Y_n}(x, y_1, ..., y_n) \\ &= f_{Y_1,...,Y_n}(y_1, y_2, ..., y_n \mid x)\, f_X(x) \\ &= f(y_1 \mid x) f(y_2 \mid x) ... f(y_n \mid x) f_X(x) \\ &= f_X(x) \prod_{i=1}^{n} f(y_i \mid x). \end{aligned}$$

The joint density of $y_1, ..., y_n$ is obtained by integrating the joint density $f_{X,Y_1,...,Y_n}(x, y_1, ..., y_n)$ with respect to the variable X .

$$f_{Y_1,...,Y_n}(y_1, ..., y_n) = \int_a^{\min_i(\frac{y_i}{\alpha_i})} f_X(x) \prod_{i=1}^{n} f(y_i \mid x)\,dx$$

$$= \frac{1}{\left(1 - b\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i}\right)\prod_{i=1}^{n}\beta_i} \times \left(\prod_{i=1}^{n} e^{-\frac{y_i - a\alpha_i}{\beta_i}} - e^{\frac{a}{b} + \min_i(\frac{y_i}{\alpha_i})\left(\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i} - \frac{1}{b}\right) - \sum_{i=1}^{n}\frac{y_i}{\beta_i}} \right)$$

Remark 5.1. *For example, substituting* $n = 2$, $a = 1$, $b = \frac{1}{2}$, $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta_1 = 3$, *and* $\beta_2 = 2$, *into the above formula, we get*:

$$f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2}\left(e^{-\frac{y_1 - 1}{3}} e^{-\frac{y_2 - 2}{2}} - e^{2 - \frac{2}{3}\min_i(\frac{y_i}{\alpha_i}) - \frac{y_1}{3} - \frac{y_2}{2}} \right) \quad (5.22)$$

that can be rewritten as

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} \frac{1}{2}\left(e^{-\frac{y_1 - 1}{3}} e^{-\frac{y_2 - 2}{2}} - e^{2 - y_1 - \frac{y_2}{2}} \right), \\ \text{if } 1 < y_1 < \frac{y_2}{2}, \quad y_2 > 2 \\ \frac{1}{2}\left(e^{-\frac{y_1 - 1}{3}} e^{-\frac{y_2 - 2}{2}} - e^{2 - \frac{y_1}{3} - \frac{5}{6} y_2} \right), \\ \text{if } 2 < y_2 < 2y_1, \quad y_1 > 1 \end{cases}$$

$f_{Y_1,Y_2}(y_1, y_2)$ integrates to 1 so it's a legitimate distribution.

Using the density of Y_i defined by (3.9) and by analogy with theorem 2.3, the expression of the m^{th} moments of Y_i is

$$E(Y_i^m) = \frac{1}{\beta_i - \alpha_i b}\sum_{k=0}^{m} k! \binom{m}{k} (a\alpha_i)^{m-k} \left(\beta_i^{k+1} - (b\alpha_i)^{k+1} \right). \quad (5.23)$$

Remark 5.2. *From* (3.11) *we deduce that for all* $i = 1, 2, ..., n$

1. $E(Y_i) = \alpha_i(a + b) + \beta_i$.
2. $V(Y_i) = \beta_i^2 + \alpha_i^2 b^2$.

The covariance between X and Y_i for $i = 1, ..., n$ is derived as:

$$\begin{aligned} &Cov(X, Y_i) = E(XY_i) - E(X)E(Y_i) \\ &= E\left(XE(Y_i \mid X)\right) - E(X)E(Y_i) \\ &= E\left(X(\alpha_i X + \beta_i)\right) - (a+b)\left(\alpha_i(a+b) + \beta_i\right) \\ &\quad \text{as } Y_i \mid X : Exp(\alpha_i x, \beta_i) \\ &= \alpha_i E(X^2) + \beta_i E(X) - \alpha_i(a+b)^2 - \beta_i(a+b) \\ &= \alpha_i b^2 + \alpha_i(a+b)^2 + \beta i(a+b) \\ &\quad - \alpha_i(a+b)^2 - \beta_i(a+b) \\ &= \alpha_i b^2. \end{aligned} \quad (5.24)$$

Bivariate case will reduce to equation (2.8).

6. Conclusion

Unlike the bivariate exponential with conditional exponential [3], and the bivariate distribution with normal conditional [4], the bivariate exponential distribution with $Exp(a, b)$ conditional has the great advantage of giving us explicit, consistent, unbiased and asymptotically unbiased estimators of our parameters *a*, *b* and *c* with reliable confidence intervals for them.

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