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Common Fixed Points for Four Self-Mappings in Dislocated Metric Space

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Abstract In this paper, we study a unique common fixed point theorem for four self mappings in dislocated metric spaces, which generalizes, extends and improves some of the recent results existing in the literature.

Keywords: *dislocated metric space, weakly compatible maps, fixed point, common fixed point*

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1. Introduction

In 2000, Hitzler and Seda [2] have introduced the notion of dislocated metric space in which self distance of a point need not be equal to zero and also generalized the Banach contraction principle in this dislocated metric space. Later on some of the authors like Aage, Salunke [1], sufati [3] and Shrivastava et.al., [5] have proved some fixed point theorems in dislocated metric space. In 2012, Jha and Panti [4] have proved some fixed point theorems for two pairs of weakly compatible maps in dislocated metric space. In this paper, we study a unique common fixed point theorem for four self mappings in dislocated metric space, which generalizes, extends and improves some known results existing in the references.

2. Preliminaries

The following definitions are due to Hitzler and Seda [2] .

Definition 2.1 [2]. Let X be a non-empty set and let d: X × X→ [0,∞) be a function satisfying the following conditions

(i) d(x, y) = d(y,x).

(ii) d(x, y) = d(y, x) = 0 ⇒ x = y.

(iii) d(x, y) ≤ d(x, z) + d(z, y) for all x, y, z ϵ X.

Then d is called dislocated metric or d-metric on X.

Definition 2.2 [2]. A sequence $\{x_n\}$ in a d-metric space (X, d) is called a Cauchy sequence if for given ∈ > 0 , there exists n_0 ∈ℕ such that for all m, n ≥ 0, we have $d(x_m, x_n) < \epsilon$.

Definition 2.3 [2]. A sequence $\{x_n\}$ in a d-metric space (X, d) converges with respect to d if there exists x ∈ X such that $d(x_n, x) \to 0$ as n→∞.

Definition 2.4 [2]. A d-metric space (X, d) is called complete if every Cauchy sequence is convergent with respect to d.

Definition 2.5 [2]. Let T and S be mappings from a metric space (X, d) itself. Then T and S are said to be weakly compatible if they commute at their coincidence point, that is, Tx = Sx for some x∈X ⇒ TSx = STx.

3. Main Results

Theorem 3.1. Let (X, d) be a complete d-metric space. Suppose S,T, I and J: X→X are continuous mappings satisfying :

$$d(Sx,Ty) \le a_1 d(Ix,Jy) + a_2 d(Ix,Sx) + a_3 d(Jy,Ty) + a_4 d(Ix,Ty) + a_5 d(Jy,Sx) \quad (1)$$

for all x, y ϵ X, where $a_i \ge 0$ (i = 1,2,3,4,5), $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$.

If S(X) ⊆ J(X), T(X) ⊆ I(X), and if the pairs (S, I) and (T, J) are weakly compatible then S,T, I and J have unique common fixed point.

Proof. Let x_0 be an arbitrary point in X.

Since S(X)⊆ J(X), T(X)⊆I(X) there exists x_1, x_2 ϵ X Such that $Sx_0 = Jx_1$, $Tx_1 = Ix_2$. Continuing this process, we define $\{x_n\}$ by $Jx_{2n+1} = Sx_{2n}$, $Ix_{2n+2} = Tx_{2n+1}$, n = 0,1,2….. Denote $y_{2n} = Jx_{2n+1} = Sx_{2n}$, $y_{2n+1} = Ix_{2n+2} = Tx_{2n+1}$, n = 0,1,2…..

If $y_{2n} = y_{2n+1}$ for some n, then $Jx_{2n+1} = Tx_{2n+1}$. Therefore, x_{2n+1} is a coincidence point of J and T. Also if $y_{2n+1} = y_{2n+2}$ for some n, then $Ix_{2n+2} = Sx_{2n+2}$. Therefore, x_{2n+2} is a coincidence point of I and S. Assume that If $y_{2n} \neq y_{2n+1}$ for all n. Then we have,

$$\begin{aligned} d(y_{2n},y_{2n+1}) &= d(Sx_{2n},Tx_{2n+1}) \\ &\le a_1 d(Ix_{2n},Jx_{2n+1}) + a_2 d(Ix_{2n},Sx_{2n}) \\ &+ a_3 d(Jx_{2n+1},Tx_{2n+1}) + a_4 d(Ix_{2n},T_{2n+1}) \\ &+ a_5 d(Jx_{2n+1},Sx_{2n}) \\ &\le a_1 d(Ix_{2n},Jx_{2n+1}) + a_2 d(Ix_{2n},Sx_{2n}) \\ &+ a_3 d(Jx_{2n+1},Tx_{2n+1}) + a_4 d(Ix_{2n},T_{2n+1}) \\ &+ a_5 d(Jx_{2n+1},Sx_{2n}) \end{aligned}$$

$$\leq a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n-1}, y_{2n})$$
$$+a_3 d(y_{2n}, y_{2n+1}) + a_4 d(y_{2n-1}, y_{2n+1})$$
$$+a_5 d(y_{2n}, y_{2n})$$
$$\leq a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n-1}, y_{2n})$$
$$+a_3 d(y_{2n}, y_{2n+1}) + a_4 [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]$$
$$+a_5 \left[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n})\right]$$
$$\leq (a_1 + a_2 + a_4) d(y_{2n-1}, y_{2n})$$
$$+(a_3 + a_4 + 2a_5) d(y_{2n}, y_{2n+1})$$

$$\Rightarrow 1-(a_3 + a_4 + 2a_5) d(y_{2n}, y_{2n+1})$$
$$\leq (a_1 + a_2 + a_4) d(y_{2n-1}, y_{2n})$$
$$\Rightarrow d(y_{2n}, y_{2n+1})$$
$$\leq (a_1 + a_2 + a_4)/1-(a_3 + a_4 + 2a_5) d(y_{2n-1}, y_{2n}).$$

Letting, $b = (a_1 + a_2 + a_4)/1\text{-}(a_3 + a_4 + 2a_5) < 1$.

$$\Rightarrow d(y_{2n}, y_{2n+1}) \leq b d(y_{2n-1}, y_{2n}).$$

This shows that

$$d(y_n, y_{n+1}) \leq b d(y_{n-1}, y_n) \leq \ldots \leq b^n d(y_0, y_1).$$

For every integer $m > 0$, we have

$$d(y_n, y_{n+m})$$
$$\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots + d(y_{n+m-1}, y_{n+m}),$$
$$\leq \left(1 + b + b^2 + \ldots + b^{n-1}\right) d(y_0, y_1),$$
$$\leq \left(b^n / 1 - b\right) d(y_0, y_1) \to 0, \text{ as } n \to \infty.$$

Therefore, $d(y_n, y_{n+m}) \to 0$.

$\Rightarrow \{y_n\}$ is a Cauchy sequence in a complete d-metric space. So there exists a point $z \in X$ such that $y_n \to z$. Therefore, the subsequences $\{Sx_{2n}\} \to z$, $\{Jx_{2n+1}\} \to z$, $\{Tx_{2n+1}\} \to z$ and $\{Ix_{2n+2}\} \to z$. Since, $T(X) \subseteq I(X)$, there exists a point $u \in X$ such that $z = Iu$. Then we have by (1)

$$d(Su, z) = d(Su, Tx_{2n+1}) + d(Tx_{2n+1}, z)$$
$$\leq a_1 d(Iu, Jx_{2n+1}) + a_2 d(Iu, Su)$$
$$+a_3 d(Jx_{2n+1}, Tx_{2n+1}) + a_4 d(Iu, T_{2n+1})$$
$$+a_5 d(Jx_{2n+1}, Su) + d(Tx_{2n+1}, z).$$

Letting $n \to \infty$ we get that

$$d(Su, z) \leq a_1 d(z, z) + a_2 d(z, Su) + a_3 d(z, z)$$
$$+ a_4 d(z, z) + a_5 d(z, Su) + d(z, z).$$
$$= (a_1 + a_3 + a_4 + 1) d(z, z) + (a_2 + a_5) d(z, Su)$$
$$\leq 2(1 + a_1 + a_3 + a_4) d(z, Su) + (a_2 + a_5) d(z, Su)$$
$$\leq (2 + 2a_1 + a_2 + 2a_3 + 2a_4 + a_5) d(z, Su),$$

which is a contradiction. So $Su = z = Iu$. Since, $S(X) \subseteq J(X)$, there exists a point $v \in X$ such that $z = Jv$.

We claim that $z = Tv$. If $z \neq Tv$. Then

$$d(z, Tv) = d(Su, Tv)$$
$$\leq a_1 d(Iu, Jv) + a_2 d(Iu, Su) + a_3 d(Jv, Tv)$$
$$+a_4 d(Iu, Tv) + a_5 d(Jv, Su)$$
$$\leq a_1 d(z, z) + a_2 d(z, z) + a_3 d(z, Tv)$$
$$+ a_4 d(z, Tv) + a_5 d(z, Su)$$
$$= (a_1 + a_2 + a_5) d(z, z) + (a_3 + a_4) d(z, Tv)$$
$$+ a_4 d(z, Tv) + a_5 d(z, Su)$$
$$\leq 2(a_1 + a_2 + a_5) d(z, Tv) + (a_2 + a_5) d(z, Su)$$
$$\leq (2 + 2a_1 + a_2 + 2a_3 + 2a_4 + a_5) d(z, Tv),$$

which is a contradiction. So we get that $z = Tv$.

Therefore, $Su = Iu = Tv = Jv = z$. That is z is a common fixed point of S, T, f and g.

Finally in order to prove that the uniqueness of z. Suppose that z and z_1, $z \neq z_1$, are common fixed points of S, T, f and g respectively. Then by (1), we have

$$d(z, z_1) = d(Sz, Tz_1)$$
$$\leq a_1 d(Iz, Jz_1) + a_2 d(Iz, Sz) + a_3 d(Jz_1, Tz_1)$$
$$+ a_4 d(Iz, Tz_1) + a_5 d(Jz_1, Sz)$$
$$\leq a_1 d(z, z_1) + a_2 d(z, z) + a_3 d(z_1, z_1)$$
$$+ a_4 d(z, z_1) + a_5 d(z_1, z)$$
$$\leq (a_1 + a_4 + a_5) d(z_1, z),$$

which is a contradiction, since $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$. Therefore, $z = z_1$.

Hence, z is the unique common fixed point of S, T, f and g respectively.

Remark 3.2. If we choose $f = g = I$ is an identity mapping in the above Theorem3.1, then we get the following corollary.

Corollary 3.3. Let (X, d) a complete d-metric space. Let S, T: X→X be continuous mappings satisfying the following

$$d(Sx, Ty) \leq a_1 d(x, y) + a_2 d(x, Sx)$$
$$+a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Sx)$$

for all x, $y \in X$, where $a_i \geq 0$ ($i = 1,2,3,4,5$), $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$.

Then S, and T have unique common fixed point.

Remark 3.4. If we choose $S = T$ in the above Theorem 3.1, then we get the following corollary

Corollary 3.5. Let (X, d) a complete d-metric space. Let S, T: X→X be continuous mappings satisfying the following

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty)$$
$$+ a_4 d(x, Ty) + a_5 d(y, Tx)$$

for all x, $y \in X$, where $a_i \geq 0$ ($i = 1,2,3,4,5$), $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$.

Then T has a unique common fixed point.

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