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Studying the Winger's "Enigma" about the Unreasonable Effectiveness of Mathematics in the Natural Sciences

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Abstract The effectiveness of mathematics in the natural sciences was characterized by the famous Nobel prize holder E. P. Winger as being unreasonable. It is not difficult for one to understand that this characterization is related to a question that has occupied the interest of philosophers, mathematicians and other scientists at least from the Plato's era in ancient Greece, until today: "Is mathematics discovered or invented by humans"? In the present work in an effort to obtain a convincing explanation of the above Winger's "enigma", the existing philosophical views about the above question are critically examined and discussed in connection with the advances in the history of mathematics that affected the human beliefs about them.

Keywords: *philosophy of mathematics, platonism, mathematical realism, non euclidean geometries, set theory, continuum hypothesis, axiom of choice, incompleteness theorems, canonical distribution, metaphysics of quality*

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1. Introduction

The present author's first research attempts during the early 1980's were focused on Pure Mathematics, a topic where one is doing research in mathematics for mathematics only, without having in mind any particular applications. The topic of his Ph.D. thesis [1] was the "Iterated Skew Polynomial Rings" (ISPR) and on that time nobody was even suspecting that this topic could find practical applications in future.

Therefore, the surprise was great, when recently the ISPR found two very important applications resulting to the renewal of the researchers' interest about them. The former concerns the ascertainment

that many Quantum Groups - i.e. Hopf algebras having in addition a structure analogous to that of a Lee group [2] - which are used as a basic tool in Theoretical Physics, can be expressed and studied in the form of an ISPR. The latter is the utilization of ISPRs in Cryptography for analyzing the structure of certain convolutional codes [3].

However, as we shall see later in this paper, analogous phenomena appear frequently in the history of mathematics, giving rise to a question that has occupied the interest of philosophers, mathematicians and other scientists for centuries: Is mathematics a ***discovery*** or an ***invention*** of the human mind?

It is not difficult for one to understand, that this question is connected to a phenomenon yhat it has been termed by the Nobelist E. P. Winger [4] as the ***unreasonable effectiveness of mathematics in the natural sciences***. Many people have weighted in on the above Winger's "enigma", notably Hilary Putman [5], Richard Hamming [6], etc. The famous astrophysicist of the Space Telescope Institute in Baltimore, USA, and best selling author Mario Livio in his book "Is God a Mathematician?" [7], which was our main source for writing this article, explores mathematical ideas from the time of Pythagoras to the present days showing us how ingenious thoughts have led to ever deeper insights into our world.

Personally, observing that the architecture of the atoms is transferred to the planets rotating around their stars, to the starts rotating around the centres of their galaxies and finally to the galaxies rotating around the centre of the whole Universe, I am thinking of the Universe as being an enormous "body", in an analogy to the human bodies. Under this consideration, the "brains" of this "body" corresponds to the "Highest Power" that has created it, which is expressed in the several human religions by the notion of the God.

In this work we try to give an explanation of the Winger's "enigma" about the success of mathematics in explaining the natural phenomena, i.e. in other words its success in explaining the architecture of the Universe. The rest of the paper is organized as follows: In the next Section we discuss the basic ideas of ***Platonism*** in the more general context of ***mathematical realism***, considering mathematics as a human discovery. In the third Section we refer to some radical advances in the history of mathematics that affected the human beliefs about its nature, including the development of the ***non Euclidean Geometries***, the axiomatic foundation of the ***Set Theory,*** the Gödel's ***incompleteness theorems,*** etc. The fourth Section discusses the opposite to the Platonism consideration of mathematics as a human invention, as well as some other ideas putting the truth somewhere in the middle between those two extreme philosophical views about its nature. Finally, the fifth Section is devoted to our conclusions.

2. Platonism and Mathematical Realism

It is well known that the success of mathematics in the natural sciences appears in two forms, which are termed by Livio [7] as the ***energetic*** and the ***pathetic*** one respectively. In the former case scientists express the laws of nature mathematically by using relations and equations developed for this purpose. The effectiveness of mathematics in this case does not look so surprising, because the relative mathematical theories are designed to fit to the corresponding observations. On the contrary, the effectiveness of the pathetic form, an example of which connected to our personal experiences was already presented in the Introduction of this paper, is really amazing. In this case completely abstract mathematical theories, developed without any intention to be applied in real life situations, are utilized in unsuspicious time for the construction of physical models!

Knot Theory, initiated from a false model for the description of the atom's structure, provides an amazing example of the interaction between the energetic and pathetic side of mathematics. In fact, the effort of mathematicians to understand the knots themselves, led eventually to the conclusion that their theory was the key for understanding the basic mechanisms of the DNA!

Another characteristic example is the use by Einstein of the ***Riemann's non Euclidean Geometry*** (see more details in the next Section) for developing the ***General Relativity Theory.*** This made Einstein to wonder: "How is it possible for mathematics, a derivative of the human mind independent from our experiences, to fit so eminently to the natural reality?".

However, this feeling of surprise is not so recent. Pythagoras and Plato felt already surprised in their distant era of the ancient Greece due to the obvious ability of mathematics to interpret the Universe. This gave to Plato the impulsion to introduce the idea of the existence of the ***universe of mathematical forms***, which probably was derived from the Pythagoreans, who believed that the Universe was totally created by the natural numbers. According to Plato, this abstract universe contains all mathematical entities (numbers, definitions, axioms, theorems, geometric figures, etc.), which are eternal and remain unchanged through the time. Consequently, humans ***do not invent*** mathematics, but they gradually ***discover*** it.

The above Plato's philosophical consideration, epigrammatically termed in our days as ***Platonism,*** despite to the blows received by the development of certain mathematical theories and to the opposite views that have been stated by many mathematicians, philosophers, cognitive scientists, psychologists, etc., it still keeps a great number of supporters. The famous in the era between the two World Wars British mathematician G.H. Hardy, for example, in his famous book "The Apology of a Mathematician" [8] writes: "I believe that mathematical reality is out of us and that our function is to discover or observe it. The mathematical theorems, which we prove considering them proudly as our own creatures, are simply the notes of our observations".

More recent views related to several variations of Platonism have been epigrammatically termed as ***Neo-Platonism*** [9].

In a more general context, all those who believe that mathematics exists independently from the human mind belong to the school of ***mathematical realism*** and they are divided into several categories with respect to their beliefs about the texture of the mathematical entities and the way in which we learn them [9].

As seen above, Platonists believe that mathematics "lives" in the eternal and unchanged universe of mathematical forms. Another view of the mathematical realism supports that mathematics is actually a piece of the natural world. The leader of this view is the cosmologist Max Tegmark, professor at MIT, who claims [10] that the Universe is not simply described by mathematics, but IT IS mathematics! His argument is based on the ambiguous hypothesis that there the natural reality is completely independent from humans. Therefore, the description of this reality ought to be released from human characteristics, like the language. Consequently, the authentic theory about the nature of the Universe must include abstract meanings and the existing relations among them only, which actually coincides with the definition of mathematics its self!.

If the above Tegmark's radical consideration about the Universe was true, it would give a very good explanation to the Winger's [4] enigma about the unreasonable effectiveness of mathematics in the natural sciences. In fact, in a Universe which is practically identified with mathematics, the fact that mathematics fits to the nature as a glove would not be surprising at all.

However, there exist some serious objections about the truth of the Tegmark's argument. Livio ([7], Chapter 9) notes that Tegmark, in order to support his view, assumes, without proving it, that mathematics is not a human invention. Also, the neuro-biologist Changeux, Professor at the College de France, states for an analogous case in Biology that it is not possible for an internal to our cognition natural situation [i.e. mathematics] to represent another natural situation external to it [i.e. the Universe] [11].

3. Radical Advances of Mathematics that Affected Human Beliefs about its Nature

It is well known that Euclid in his "Elements" (300 B. C.) created the theoretical framework of the traditional Geometry based on 10 axioms, which were used to prove the several geometric propositions. The fifth of those axioms, stated in its present form [1] by the Greek

[1] From a point of the plane lying outside a given straight line only one parallel can be drawn to this line. According to the original Euclid's statement, two straight lines of the plane intersecting a third one such that the two internal angles formed in the one side have sum less than two right angles, then the two lines, when extended, are always intersected from this side.

mathematician Proclus in his comments on the "Elements" (5th century A. C.), created during the centuries many objections among mathematicians, because it has not the plainness of the rest of the Euclid's axioms [2]. Many attempts have been made to prove it with the help of the other nine axioms, or to replace it with another, more obvious axiom. When all these attempts definitely failed, the question "what if it is not true" occupied the interest of the researchers of mathematics.

The Russian mathematician Nikolai Ivanovich Lobachevsky (1792-1856) was the first who replaced this axiom with the statement that in a plane and from a point not belonging to a given straight line it is possible to draw ***at least two*** straight lines which are parallel to the given one. Accordingly a new Geometry was created, the ***Hyperbolic Geometry*** which is developed on a hyperbolic paraboloid (saddle surface). Note that in this type of Geometry the sum of the angles of a triangle is always less than 180^o (Figure 1).

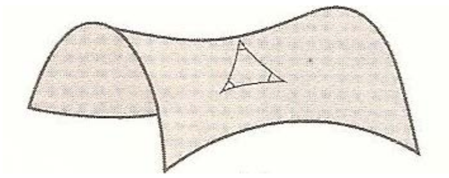

Figure 1. Hyperbolic Paraboloid

A similar Geometry was introduced independently by the young Hungarian mathematician Janos Bolyai (1802-1860). Analogous ideas where also stated, independently to each other, by the great German mathematician Carl Friedrich Gauss (1777-1855) and by the Professor of Law Ferdinand Swickard (1780-1859). However, both of them never decided to publish their ideas. The reason was probably the fact that in their era, which was dominated by the Immanuel Kant's (1724-1804) philosophical belief that the Euclidean Geometry is an absolute truth [12], such kind of ideas could be considered as a philosophical heresy.

Nevertheless, Bernhard Riemann (1826-1866), one of the Gauss's students, in a historic lecture performed at the University of Cottingen on June 10th, 1854 [3] showed that the Hyperbolic Geometry is not the only possible non Euclidean Geometry and he introduced the ***Elliptic Geometry***, which is developed on a sphere's surface (Figure 2). It can be easily understood that in this type of Geometry all curves, including straight lines, are closed, while the shortest path between two points is the smaller arrow of a great circle of the sphere defined by those points. Also, ***no parallel*** to a given line can be drawn from a point outside of it, as it happens for example with any two meridians which, although they look like being parallel near the equator, they finally meet at the two poles of the sphere. Further, the sum of the angles of a triangle is always greater than 180^o. Obviously, in a small distance around an observer, the Euclidean arises as a special case of the Elliptic Geometry.

[2] All indications show that even Euclid himself was not completely satisfied with this axiom, which is not used in the proofs of the first 28 propositions in his "Elements".

[3] The title of the lecture was "On the hypotheses which underline Geometry". An English translation of this lecture can be found in [14].

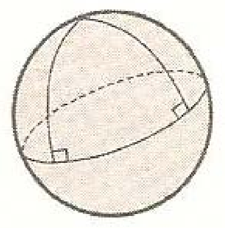

Figure 2. Framework of the Elliptic Geometry

Riemann advanced further his ideas by defining analogous Geometries in spaces with more than three dimensions [4]. In particular, Einstein's ***General Relativity Theory*** was developed by using the Riemann's principles in a 4-dimensional space with its fourth dimension corresponding to time. Riemann gave also an accurate definition of the curvature of a curve or a surface. His idea was to introduce a collection of numbers at every point in space, known as the ***curvature's tensor***, which would describe how much the curve or the surface was bent or curved. This idea gave genesis to Differential Geometry that connected Geometry with Mathematical Analysis, as the Descartes Analytic Geometry had done before with Geometry and Linear Algebra.

The development of the non Euclidean Geometries caused a real sock to the Platonists, because the Euclidean Geometry was considered until that time as the strongest component of their abstract and unchanged "universe of mathematics". Since then many people began to suspect that mathematics could finally be a human invention rather than a discovery.

However, a greater sock followed after a while, connected to the development of the ***Set Theory*** by Georg Cantor (!845 - 1918). In fact, the paradoxes appeared in this theory between the end of the 19th and the beginning of the 20th century [9], gave the reason to the German mathematician Emst Zermelo (1871-1953), following the road opened by Euclid for Geometry many centuries ago, to suggest in 1908 a way of restating the Set Theory in terms of a system of axioms. As a result, the paradoxes were by-passed through a careful statement of the corresponding axioms so that to blockade contradictory notions like "the set of all the sets", which happens to be a member of its self (***Russell's paradox***).

The axiomatic system of Zermelo was enriched by Fraenkel in 1922 and was further improved by Von Neumann in 1925, so that everything seemed to work well. But gradually one of those axioms started to cause headache to the mathematicians. This was ***the axiom of choice*** stating that, if X is a set of non empty sets, then one can choose a unique element from each of these sets in order to create a new set Y.

When X is a finite set, or when it is an infinite set but we know the rule under which the choice is made, then the above statement is obvious. The problem is located when X is an infinite set and the rule of the choice is unknown. In this case the choice does never end and the existence of Y becomes a matter of faith. For example, assuming that

[4] Spaces with more than three dimensions were first introduced by the German teacher Hermann Grassmann (1809-1877), a polymath lacking university studies in mathematics. His ideas, published in his book entitled "Theory of Linear Extension: A New Branch of Mathematics" [13] gave genesis to the Linear Algebra.

X is an infinite set of pairs of shoes, if we decide to choose always the right shoe from each pair, then there is no problem. On the contrary, if X is an infinite set of pairs of stockings, then obviously we have a problem with the choice.

This disadvantage made the mathematicians to start thinking, as it had happened with the fifth Euclid's axiom, if the axiom of choice could be either proved or by-passed with the help of the other axioms. The answer to this question was partially given by Curt Gödel [15], who proved that the axiom of choice as well as the Cantor's ***continuum hypothesis*** [5] are consistent to the other Zermelo-Fraenkel axioms; i.e. they cannot be contradicted by them. In particular, for the continuum hypothesis this remains true even if the axiom of choice is added to the other Zermelo-Fraenkel axioms.

The Gödel's result was completed by the American mathematician Paul Cohen (1934-2007), who proved in 1963 that the axiom of choice and the continuum hypothesis cannot be proved by the other axioms of Set Theory and that this is true for the continuum hypothesis even if the axiom of choice is added to the other axioms [16]. The Cohen's result combined with the Gödel's outcomes, shows that the axiom of choice and the continuum hypothesis are independent from the other axioms of Set Theory. Therefore, considering the continuum hypothesis as an axiom and adding it to the system of the Zermelo-Fraenkel axioms, one can create four different theories for the Sets: The first one by including to it both the axioms of choice and the continuum, the next two by including only one of them in each case and the fourth one by including none of them!

The fundamental role of the Set Theory for the whole spectre of mathematics made this new sock more intense for the Platonists, although they didn't seem to retreat from their positions claiming that the four different Set Theories pre-existed in the universe of mathematics. Gödel himself, who believed that the mathematical truth is indeed independent from humans, in an article published in 1947 wrote that there exists a kind of mathematical intuition that makes one to catch directly the mathematical notions in a way analogous to the natural notions.

It must be underlined here that Gödel (1906-1978) became widely known mainly for his two ***incompleteness theorems*** [17] published in 1931, one year after getting his Ph.D. degree based on them. These theorems state that, if S is a consistent formal system consisting of a finite set of axioms and rules then:

1. S is incomplete, i.e. there exist propositions which cannot be either proved or refuted inside S , and
2. Its consistency cannot be proved inside S.

The incompleteness theorems caused a strong earthquake to the mathematical and philosophical world, putting, among others, a definite end to the program of the leader of ***formalism*** David Hilbert (1862-1943) about a complete and consistent - i.e. not permitting the creation of absurd situations - axiomatic development of all branches of mathematics [9]. In fact, according to the second theorem there is no system that can prove the consistency of another system, since it has first to prove its own consistency! Therefore, the best to hope is that the statement of a certain system's axioms, although it cannot be complete (first theorem), it is consistent.

It must be clarified that the incompleteness theorems do not imply that some truths will never be known, neither that the human ability of understanding is in a way restricted, but they simply underline the weaknesses and deficiencies of the typical systems.

It was normally expected that the proof of the incompleteness theorems would stagger the people's confidence for the effectiveness of mathematics, but the opposite was actually happened! In fact, the amazing success of mathematics in the natural theories about the Universe the decades before and after the publication of the Gödel's theorems, supported by the introduction of new mathematical branches, like Operational Research, Fractals, Non Linear Dynamics, Fuzzy Logic, etc., made mathematics to be more and more essential for the understanding not of the real world only, but also of almost all human activities.

For example, the Belgian mathematician and astronomer Adolphe Quetelet (1796-1874) was among the first to apply Statistics to Social Sciences planning, creating what he called the ***Social Physics***. Among others, he introduced the concept of the ***average man*** characterized by the mean values of measure variables that follow the ***Gauss's normal distribution***. He collected data about many such variables like crime, marriage and suicide rates, biological and spiritual human characteristics, etc.

It is recalled that the graph of a normal distribution is a curve having the form of a church-bell, which is symmetric around its mean value μ (Figure 3).

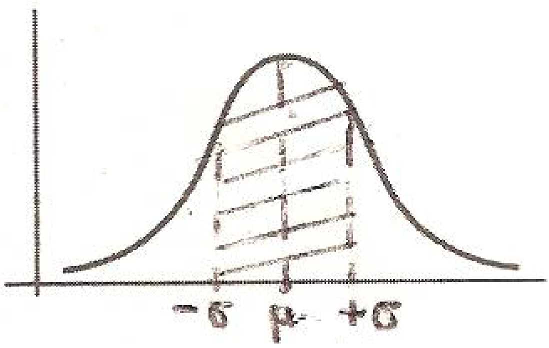

Figure 3. Graph of a normal distribution

Before Quetelet's observations this curve was known as "the errors' curve", because it was observed that it appears in all kinds of errors around the mean values in astronomical measurements. Using the known method of the Integral Calculus for calculating the area under a curve it can be shown that in a normal distribution 68.2 % of the corresponding data take values in the area defined by one standard deviation σ in each side of the mean value μ (Figure 3). For example, if the mean value of the heights of a large enough number of people is 170 cm and the standard deviation is 20, then 68.2% of those people have

[5] The continuum hypothesis, which was the first of the 23 unsolved mathematical problems presented in 1900 by Hilbert at the International Conference of Mathematics in Paris, states that the set of real numbers has the minimal possible cardinality which is greater than the cardinality of the set of non negative integers. This is equivalent of saying that the cardinality of the power set of the non negative integers is equal to the cardinality of the real numbers. In an analogous way, the ***generalized continuum hypothesis*** states that the cardinality of the power set of each infinite set is the smaller cardinality which is greater than the cardinality of this set.

heights between 150 and 190 cm. Also, the probability of a person to have a height between 170 and 190 cm is 38.1%. In other words, in this case Probability and Statistics are completing each other.

4. The Consideration of Mathematics as a Human Invention and Other Intermediate Theories

In contrast to Platonists, many other people believe that mathematics is a human invention. Kasser & Newman [18], for example, note: "It looks strange to us that such a theory [Platonism] could ever exist…… The non Euclidean Geometries provide now a strong proof that mathematics is a creature of the human mind, being only under the restrictions imposed by the laws of reasoning".

Cognitive scientists and psychologists have produced in our days a series of arguments supporting the correlation of mathematics with the human mind. These arguments are mainly based on the data of many experiments concerning the neuro-mapping of the human brains during the performance of proper mathematical activities. An attributive statement from these sources comes from Lakoff and Nunez [19] claiming that mathematics was created by the humans, who are also responsible for its preservation and further extension.

Nevertheless, the most convincing argument probably comes from Sir Michael Atiyah [20], one of the top mathematicians of the 20th century, having as follows: The mathematical notion with the greater probability to exist independently from human mind is the natural numbers. Even Leopold Kronecker (1823-1891), one of the main supporters of ***intuitionism,*** used to say that "God created the natural numbers, while everything else was created by humans" [9]. Let us now imagine that there exists a jellyfish having the ability of thinking logically, which is living completely isolated in the depths of the ocean. In such a pure continuum there is nothing to be measured, therefore it is impossible for the jellyfish to INVENT by abstraction the notion of the natural numbers.

One could of course disagree with the use of the hypothetical universe of the jellyfish that supports the above argument, claiming that each hypothesis must be examined within the existing real Universe. However, such a claim is actually equivalent of accepting that the notion of natural numbers depends on the human experiences and therefore it is a human invention**!**

Although impressive, Sir Atiyah's argument could be opposed by a Platonist by claiming that under those conditions it would in fact be impossible for the jellyfish to DISCOVER the natural numbers, but this does not mean that this notion does not exist in the eternal and unchanged "universe of mathematical forms"!

Sir Atiyah [20] also notes that, since the human brains was developed in order to confront the natural world, it is not so surprising that it created mathematics in a way compatible to this purpose. This explains in part the unreasonable effectiveness of mathematics in the natural sciences, although it leaves some doubts about the explanation of the "pathetic" form [7] of this effectiveness.

On the other hand, views have been also appeared putting the truth somewhere in the middle between the two extreme theories that mathematics are discovered or invented. Livio ([7], Chapter 9) for instance, claims that mathematics is ***a mixture of human discoveries and inventions***. The axioms and definitions are inventions, like it happens with the rules of chess, while the related theorems are discoveries. In many cases theorems are created by their proofs, i.e. the mathematicians study first what they can prove, wherefrom they derive the theorems. In other cases, like the ancient Greek mathematician and engineer Archimedes describes in his "Method", the answer to a problem can be firstly found intuitively or empirically and the proof follows.

As an example, Livio [7] refers to the prime numbers, which as a notion is an invention of the ancient Greek mathematicians, while all the theorems related to them are human discoveries. The ancient Babylonians, Egyptians and Chinese never used prime numbers in their mathematics, but it is not logical to consider that they didn't discover them. They simply managed to progress their mathematics without them, like the United Kingdom is governed without a written Constitution!

Nicholson [23] invokes the introduced by Pirsig [24,25] philosophical system of the "***Metaphysics of Quality***" to support, a belief analogous to the above Livio's view and to give an explanation of the Winger's enigma about the unreasonable effectiveness of mathematics in the natural sciences.

Before exposing the Nicholson's ideas, we recall first that it has been claimed in many different ways by many people and especially by working mathematicians that mathematics follows quality. One of the best arguments about this comes from the famous Princeton mathematician Goro Shimura. Discussing his famous Taniyama – Shimura conjecture, whose proof by Andrew Wiles led to the proof of the Fermat's last Theorem [21] Shimura stated: "Mathematics should contain goodness….. It is a rather crude philosophy, but one can always take it as a starting point….. I might say that the conjecture stemmed from that philosophy of goodness. Most mathematicians do mathematics from an aesthetic point of view and the philosophy of goodness comes from my aesthetic viewpoint" ([22], p.210).

According to Pirsig [25] "quality" is a non definable entity that can be understood by splitting it to the "***dynamic***" and to the "***static***" quality, which they act like a ratchet: The dynamic quality is the constant stimulus to move to something better, to ratchet up, whereas the static quality is the latch of the ratchet itself, the making tangible of the motion up into something concrete, which will prevent falling down into something worse. In other words, the dynamic quality is the creative urge, whereas the static quality is what is created in response.

Nicholson [23] argues that mathematics is invented insofar as it is a process following dynamic quality and it is discovered insofar as it is a process of slashing out previously unknown consequences within static patterns of quality that are mathematics as it stands. Further, his explanation about the Winger's enigma is based to the argument that, since nature and mathematics are both patterns of static quality created by following dynamic quality, it is not surprising that they arrive to the same conclusions.

We shall close this discussion with an interesting remark of Davis & Hersh contained in their book "The Mathematical Experience" [26]: "The middle working mathematician is a Platonist during the working days of the week and a formalist (therefore it considers mathematics as a human invention) on Sundays. This means that when he/she is working on mathematics he/she is convinced that he/she is faced with an objective reality, but when he/she is asked to provide a philosophical explanation of this reality, he/she considers easier to answer that he/she does not believe it!'

5. Conclusion

From the discussion performed in this paper it turns out that there exist many indications in our days that mathematics is invented by humans. Nevertheless, the corresponding arguments have not been proved strong enough until now to oppose in a categorical way the Platonic view that mathematics is independent from the human mind and therefore it is a human discovery. Accordingly, a possible scenario is that mathematics is a mixture of human discoveries and inventions. All these scenarios are connected, to the Wigner's "enigma" of the unreasonable effectiveness of mathematics in the natural sciences, since they are trying to explain it..

However, it is of worth to notice that all the above belong to the sphere of philosophy, which focuses in general on the continuous study rather of such kind of questions, than on finding definite answers about them. A study that improves our conception about the several possibilities, enriches our spiritual imagination and decreases the dogmatic perception that restricts the power of reasoning in favour of the conjecture [27].

Adopting a Livio's remark ([7], Chapter 9), we personally believe that the only certain conclusion that can be obtained from such a discussion is that ***mathematics constitutes a part of the human civilization***. For example, the more recent European mathematicians, following the road opened by Euclid for Geometry many centuries ago, not only used the axiomatic method for the development of other branches of mathematics, like Zermelo did for the Set Theory, but they also developed the philosophical theory of formalism, whose target was – until it was stopped by the proof of the Gödel's incompleteness theorems - a general axiomatization of mathematics.

In the Chinese philosophy ***Yin*** and ***Yang*** represent all the opposite principles. Nevertheless, it is important to pay attention to the fact that these two aspects complete rather than opposing each other, with the one containing some part of the other [28]. Accordingly, each of the several philosophies of mathematics has its own importance and advantages, but what it matters more is to find a proper balance among them.

In particular, the implications that the philosophical ideas connected to mathematics have on Mathematics Education are serious and important. This is of course a theme that needs a separate and extensive analysis, which is out of the scope of the present work. Nevertheless, it is obvious that the required balance among these ideas could prevent in future the "earthquakes" in the sensitive area of teaching and learning mathematics - like it was happened in the near past with the unsuccessful introduction of the "new mathematics" in school education - that have so much distressed students and teachers during the last years.

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