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A Common Fixed Point Result in Ordered Complete Cone Metric Spaces

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Abstract In this paper, we prove a common fixed point theorem for ordered contractions in ordered cone metric spaces without using the continuity. Our result generalizes some recent results existing in the references.

Keywords: *fixed point, common fixed point, ordered cone metric space, normal cone, nonnormal cone*

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1. Introduction

In 2007, Huang and Zhang [5] introduced the concept of a cone metric space and proved some fixed point theorems in cone metric space. Later on, many authors have generalized and extended the fixed point theorems of Huang and Zhang [5]. Fixed point theorems in partially ordered set was studied by Ran and Reurings [9], Nieto and Lopez [8]. Subsequently, many authors (see, e. g., [1,2,6]) were investigated the fixed point results on ordered metric spaces. Altun and Durmaz [4], Altun , Damnjanovic and Djoric [3] obtained fixed point theorems in ordered cone metric spaces. Recently, Kadelburg, Pavlovic and Radenovic [7] proved some common fixed point theorems in ordered contractions and quasicontractions in ordered cone metric spaces. In this paper, we proved a common fixed point theorem in ordered cone metric spaces without using the continuity. Our result, generalizes the results of [7].

The following definitions are in [5].

Definition 1.1. [5] Let E be a real Banach space and P be a subset of E. The set P is called a cone if and only if:

(a). P is closed, non–empty and $P \neq \{0\}$;

(b). $a, b \in \mathbb{R}$, $a,b \geq 0$, $x,y \in P$ imply $ax+by \in P$;

(c). $x \in P$ and $-x \in P$ implies $x = 0$.

Definition 1.2.[5] Let P be a cone in a Banach space E, define partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y-x \in P$. We shall write $x \prec y$ to indicate $x \preceq y$ but $x \neq y$ while $x \ll y$ will stand for $y-x \in \operatorname{int} P$, where int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1.3.[5] Let E be a Banach space and $P \subset E$ be an order cone. The order cone P is called normal if there exists L>0 such that for all $x, y \in E$,

$$0 \preceq x \preceq y \Rightarrow \|x\| \leq \|y\|.$$

The least positive number L satisfying the above inequality is called the normal constant of P.

Most of ordered Banach spaces used in applications posses a cone with the normal constant K = 1.

Definition 1.4. [5] Let X be a nonempty set of E. Suppose that the map $d: X \times X \rightarrow E$ satisfies:

(d1). $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2). $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3). $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Remark 1.5. [7] (1) If $u \preceq v$ and $v \ll w$, then $u \ll w$.

(2) If $0 \preceq u \ll c$ for each $c \in \operatorname{int} P$, then $u = 0$.

(3) If $a \preceq b + c$ for each $c \in \operatorname{int} P$, then $a \preceq b$.

(4) If $0 \preceq x \preceq y$ and $0 \leq a$, then $0 \preceq ax \preceq ay$.

(5) If $0 \preceq x_n \preceq y_n$, for each $n \in \mathbb{N}$, and $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, then $0 \preceq x \preceq y$.

(6) If $0 \preceq d(x_n, y_n) \preceq b_n$ and $b_n \rightarrow 0$, then, $d(x_n,x) \ll c$ where x_n , x are respectively, a sequence and a given point in X.

(7) If E is a real Banach space with a cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 < \lambda < 1$, then $a = 0$.

(8) If $c \in \operatorname{int} P$, $0 \preceq a_n$ and $a_n \rightarrow 0$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$.

2. Main Result

In this section, we prove a common fixed point theorem in an ordered complete cone metric spaces.

Theorem 2.1. Let $(X, \sqsubseteq, d)$ be an ordered complete cone metric cone space. Let (f, g) be weakly increasing pair of self-maps on X w. r. t. $\sqsubseteq$. Suppose that the following conditions hold:

(i) there exists $p, q, r, s, t \geq 0$ satisfying $p + q + r + s + t < 1$ and $q = r$ or $s = t$, such that

$$d(fx,\ gy) \preceq pd(x,y) + qd(x,fx) + rd(y,gy) + sd(x,gy) + td(y,fx) \quad (1)$$

for all comparable x, y∈X;

(ii) if a nondecreasing sequence $\{x_n\}$ converges to x∈X, then $x_n \sqsubseteq x$ for all $n\in\mathbb{N}$. Then, f and g have a common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all n∈N. Since, (f, g) is weakly increasing , it can be easily shown that the sequence $\{x_n\}$ is nondecreasing w. r. t. ⊑, that is, $x_0\sqsubseteq x_1\sqsubseteq\ldots\sqsubseteq x_n\sqsubseteq x_{n+1}\sqsubseteq\ldots$. In particular, x_{2n} and x_{2n+1} are comparable, by (1) we have

$$\begin{aligned}&d(x_{2n+1},x_{2n+2}) = d(fx_{2n},gx_{2n+1})\\ &\preceq pd(x_{2n},x_{2n+1}) + qd(x_{2n},x_{2n+1})\\ &+rd(x_{2n+1},x_{2n+2}) + sd(x_{2n},x_{2n+2})\\ &+td(x_{2n+1},x_{2n+1})\\ &\preceq pd(x_{2n},x_{2n+1}) + qd(x_{2n},x_{2n+1}) + rd(x_{2n+1},x_{2n+2})\\ &+s[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})].\end{aligned}$$

It follows that

$$(1-r-s)d(x_{2n+1},x_{2n+2}) \preceq (p+q+s)d(x_{2n},x_{2n+1}).$$

That is,

$$d(x_{2n+1},x_{2n+2}) \preceq \frac{p+q+s}{1-r-s}d(x_{2n},x_{2n+1}). \qquad (2)$$

Similarly, we obtain

$$d(x_{2n+2},x_{2n+3}) \preceq \frac{p+q+t}{1-q-t}\frac{p+q+s}{1-r-s}d(x_{2n},x_{2n+1}).$$

From (1) and (2), by induction, we obtain that

$$\begin{aligned}&d(x_{2n+1},x_{2n+2}) \preceq \frac{p+q+s}{1-r-s}d(x_{2n},x_{2n+1})\\ &\preceq \frac{p+q+s}{1-r-s}.\frac{p+r+s}{1-q-t}d(x_{2n-1},x_{2n})\\ &\preceq \frac{p+q+s}{1-r-s}.\frac{p+r+s}{1-q-t}.\frac{p+r+s}{1-q-t}d(x_{2n-2},x_{2n-1})\\ &\preceq \ldots \preceq \frac{p+q+s}{1-r-s}.\left(\frac{p+r+s}{1-q-t}.\frac{p+q+s}{1-r-s}\right)^n d(x_0,x_1),\end{aligned}$$

and

$$\begin{aligned}&d(x_{2n+2},x_{2n+3}) \preceq \frac{p+q+t}{1-q-t}d(x_{2n+1},x_{2n+2})\\ &\preceq \ldots \preceq \left(\frac{p+r+s}{1-q-t}.\frac{p+q+s}{1-r-s}\right)^{n+1} d(x_0,x_1).\end{aligned}$$

Let $M = \frac{p+q+s}{1-r-s}$, $N = \frac{p+r+s}{1-q-t}$.

In the case q = r,

$$MN = \frac{p+q+s}{1-r-s}.\frac{p+r+s}{1-q-t} < 1\times 1 = 1.$$

Now, for n < m we have

$$\begin{aligned}&d(x_{2n+1},x_{2m+1}) \preceq d(x_{2n+1},x_{2n+2}) + \ldots + d(x_{2n},x_{2m+1})\\ &\preceq \left(M\sum_{i=n}^{m-1}(MN)^i + \sum_{i=n+1}^{m1}(MN)^i\right)d(x_0,x_1),\\ &\preceq \left(\frac{M(MN)^n}{1-MN} + \frac{(MN)^{n-1}}{1-MN}\right)d(x_0,x_1),\\ &= (1+N)\frac{M(MN)^n}{1-MN}d(x_0,x_1).\end{aligned}$$

Similarly, we obtain

$$d(x_{2n},x_{2n+1}) \preceq (1+M)\frac{(MN)^n}{1-MN}d(x_0,x_1),$$

$$d(x_{2n},x_{2m}) \preceq (1+M)\frac{(MN)^n}{1-MN}d(x_0,x_1),$$

and $d(x_{2n+1},x_{2m}) \preceq (1+N)\frac{M(MN)^n}{1-MN}d(x_0,x_1)$.

Hence, for n < m

$$d(x_n,x_m) \preceq \max\left\{\begin{array}{l}(1+N)\frac{M(MN)^n}{1-MN},\\ (1+M)\frac{(MN)^n}{1-MN}\end{array}\right\}d(x_0,x_1)$$

$$= b_n d(x_0,x_1),$$

where $b_n \to 0$, as n→∞.

By using (8) and (1) of Remark 1.5 and only the assumption that the underlying cone is solid, we conclude that $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is complete, there exists u∈X such that $x_n \to u$ (as n→∞).

$$\begin{aligned}&d(fu,x_{2n+2}) = d(fu,gx_{2n+1})\\ &\preceq pd(u,u) + qd(u,fu) + rd(x_{2n+1},gx_{2n+1})\\ &+sd(u,gx_{2n+1}) + td(x_{2n+1},fu).\end{aligned}$$

Letting n→+ ∞

$$\begin{aligned}(fu,u) &\preceq pd(u,u) + qd(u,fu) + rd(u,gu)\\ &+ sd(u,gu) + td(u,fu)\\ &\preceq (q+t)d(u,fu) + (r+s)d(u,gu). \qquad (3)\end{aligned}$$

$$\Rightarrow (1-q-t)d(fu,u) \preceq (r+s)d(u,gu).$$

$$\Rightarrow d(fu,u) \preceq \left(\frac{r+s}{1-q-t}\right)d(u,gu).$$

Let c >> 0 be given. Choose a natural number N_1 such that d(u, gu) << $\left(\frac{r+s}{1-q-t}\right)$ c. Then from (3) we get that d(fu, u) << c.

Since c is arbitrary, we get that

$$d(fu,u) << \frac{c}{m} \text{ for each } m \in \mathbb{N}$$

Noting that $\frac{c}{m} \to 0$ as m→∞, we conclude that $\frac{c}{m} - d(fu,u) \to d(fu,u)$ as m→∞.

Hence, P is closed, then - d(fu, u) ∈ P.

Thus d(fu, u) ∈ P ∩ (-P). Hence d(fu, u) = 0.

Therefore, fu = u.
And

$$d(fx_{2n+1},gx_{2n+2}) \preceq pd(x_{2n+1},x_{2n+2}) + qd(x_{2n+1},fx_{2n+1}) + rd(x_{2n+2},gx_{2n+2}) + sd(x_{2n+1},gx_{2n+1}) + td(x_{2n+2},fx_{2n+1})$$
$$\preceq pd(x_{2n+1},x_{2n+2}) + qd(x_{2n+1},fx_{2n+1}) + rd(x_{2n+2},gx_{2n+2}) + sd(x_{2n+1},gx_{2n+1}) + td(x_{2n+2},fx_{2n+1}).$$

Letting $n \to +\infty$

$$d(fu,gu) \preceq pd(u,u) + qd(u,fu) + rd(u,gu) + sd(u,gu) + td(u,fu)$$
$$\Rightarrow d(fu,gu) \preceq pd(u,u) + qd(u,u) + rd(fu,gu) + sd(fu,gu) + td(u,u),$$
$$\Rightarrow d(fu,gu) \preceq (r+s)d(fu,gu),$$
$$\Rightarrow (1-(r+s))d(fu,gu) \preceq 0,$$
$$\Rightarrow (1-(r+s))d(fu,gu) \preceq 0,$$
$$\Rightarrow d(fu,gu) \preceq 0.$$

That is, fu = gu.
Now we show that fu = gu = u. By (1), we have

$$d(x_{2n+1},gu) = d(fx_{2n},gu) \preceq pd(x_{2n},u) + qd(x_{2n},fx_{2n}) + rd(u,gu) + sd(x_{2n},gu) + td(u,fx_{2n}).$$

Letting $n \to +\infty$

$$d(u,gu) \preceq pd(u,u) + qd(u,fu) + rd(u,gu) + sd(u,gu) + td(u,u)$$
$$\preceq pd(u,u) + qd(u,u) + rd(u,gu) + sd(u,gu) + td(u,u)$$
$$\preceq (r+s)d(u,gu)$$
$$\Rightarrow (1-r-s)d(u,gu) \preceq 0$$
$$\Rightarrow d(u,gu) \preceq 0$$
$$\Rightarrow d(u,gu) = 0. \text{ That is, } u = gu.$$

Therefore, fu = gu = u and u is a common fixed point of f and g.

Now, we consider the case when condition (ii) is satisfied. For the sequence $\{x_n\}$ we have $x_n \to u \in X$(as $n \to \infty$) and $x_n \sqsubseteq u (n \in \mathbb{N})$. By the construction, $fx_n \to u$ and $gx_n \to u$(as $n \to \infty$).

Let us prove that u is a common fixed point of f and g. Putting x = u and y = x_n in (1)(since they are comparable) we get that

$$d(fu,gx_n) \preceq pd(u,x_n) + qd(u,fu) + rd(x_n,gx_n) + sd(u,gx_n) + td(x_n,fu).$$

For the first and fourth term of the right hand side we have $d(x_n,u) \ll c$ and $d(u, gx_n) \ll c$(for $c \in \text{int } P$ arbitrary and $n \geq n_0$). For the second term $d(u, fu) \preccurlyeq d(u, x_n) + d(x_n, gx_n) + d(gx_n, fu)$(again the first term n the right can be neglected) and for the fifth term $d(x_n, fu) \preccurlyeq d(x_n, gx_n) + d(gx_n, fu)$. It follows that

$$(1-q-t)d(fu,gx_n) \preceq (q+r+t)d(x_n,gx_n).$$

But $x_n \to u$ and $gx_n \to u \Rightarrow d(x_n, gx_n) \ll c$, which means that $d(fu, gx_n) \ll c$, that is, $gx_n \to fu$. It follows that, fu = u and in a symmetric way (by using that $u \sqsubseteq u$), gu =u.

Remark 2.2. If we choose f and g are continuous mappings in the above Theorem 2.1, then we get the Theorem 2.1 of [7].

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