

# A Boundary Value Problem for the Equation of Motion of a Homogeneous Bar with Periodic Conditions

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**Abstract** In this paper the classical solution of a nonlocal boundary value problem for the equation of motion of a homogeneous bar is investigated. Then using Fourier's method stated problem reduced to an integral equation. Further, exploiting the contracting mappings principle the existence and uniqueness of the classical solution for the considered boundary value problem is proved

**Keywords:** nonlocal, boundary value problem, classical solution, Fourier method, homogeneous bar

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## 1. Introduction

The non-local problems are the problems wherein instead of giving the values of the solution or its derivatives on the fixed part of the boundary, the relation of these values with the values of the same functions on another inner or boundary manifolds is given. Theory of non-local boundary value problems is important in itself as a section of general theory of boundary value problems for partial equations and it is important as a section of mathematics that has numerous applications in mechanics, physics, biology and other natural science disciplines.

The more general time non-local conditions were considered were considered on the papers of A.A.Kerefov, J.Chabrowsky [8], V.V.Shelukhin [9], G.M.Liberman [10], A.I.Kozhanov [11], and others.

Yu.A.Mitropolsky and B.I.Moiseenkov [1], J.M.T. Thompson, H.B. Stewart [2], B.S.Bardin, S.D.Furta [3], D.V.Kostin [4] and others have situated oscillation and wave motions of an elastic bar on an elastic foundation.

The simplest non-linear model of motion of a homogeneous bar is described by the equation

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + k \frac{\partial^2 w}{\partial x^2} + \alpha w + w^3 = 0,$$

where  $w$  is bar's deflection (after displacement of the middle line points of an elastic bar along the axis  $x$ ). Note that the similar equation arises in the theory of crystals [5].

## 2. Problem Statement and ITS Reduction to an Integral Equation

For the equation [4]

$$u_{tt}(x,t) + u_{xxxx}(x,t) + \beta u_{xx}(x,t) + \alpha u(x,t) + u^3(x,t) = 0 \quad (1)$$

in the domain  $D_T = \{(x,t) : 0 \leq x \leq 1, 0 \leq t \leq T\}$  we consider a problem with ordinary periodic boundary conditions

$$u_x(0,t) = u_x(1,t), \quad u(0,t) = u(1,t), \quad u_{xx}(0,t) = u_{xx}(1,t), \quad (2)$$

$$u_{xx}(0,t) = u_{xx}(1,t), \quad u_{xxx}(0,t) = u_{xxx}(1,t), \quad (0 \leq t \leq T),$$

and subject to the non-local boundary conditions

$$u(x,0) + \delta u(x,T) = \phi(x), \quad (3)$$

$$u_t(x,0) + \delta u_t(x,T) = \psi(x) \quad (0 \leq x \leq 1)$$

where  $\beta > 0$ ,  $\alpha > 0$ ,  $\delta$  are the given numbers, moreover  $\beta < 4\alpha$ ,  $\phi(x)$ ,  $\psi(x)$  are the given functions,  $u(x,t)$  is a sought function.

### Definition

Under the classic solution of problem (1)-(3) we understand the function  $u(x,t)$ , continuous in the closed domain  $D_T$  together with its all derivatives involved in equation (1), and satisfying all the conditions (1)-(3) in ordinary sense.

It is known [11] that the system

$$1, \cos \lambda_1 x, \sin \lambda_1 x, \dots, \cos \lambda_k x, \sin \lambda_k x, \dots$$

is a basis in  $L_2(0,1)$ , where  $\lambda_k = 2k\pi$  ( $k=1,2,\dots$ ).

Then it is obvious that each classical solution  $u(x,t)$  of problem (1)-(3) has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t) \sin \lambda_k x \quad (\lambda_k = 2k\pi), \quad (4)$$

where

$$u_{10}(t) = \int_0^1 u(x,t) dx, u_{1k}(t) = 2 \int_0^1 u(x,t) \cos \lambda_k x dx,$$

$$u_{2k}(t) = 2 \int_0^1 u(x,t) \sin \lambda_k x dx (k = 1, 2, \dots).$$

Then, applying the formal scheme of the Fourier method, from (1) and (3) we have:

$$u''_{10}(t) + \alpha u_{10}(t) = F_{10}(t; u) (0 \leq t \leq T), \tag{5}$$

$$u''_{ik}(t) + (\lambda_k^4 - \beta \lambda_k^2 + \alpha) u_{ik}(t) = F_{ik}(t; u) \tag{6}$$

$$(0 \leq t \leq T; i = 1, 2; k = 1, 2, \dots)$$

$$u_{10}(0) + \delta u_{10}(T) = \phi_{10}, \tag{7}$$

$$u'_{10}(0) + \delta u'_{10}(T) = \psi_{10},$$

$$u_{ik}(0) + \delta u_{ik}(T) = \phi_{ik}, u'_{ik}(0) + \delta u'_{ik}(T) = \psi_{ik}, \tag{8}$$

$$(i = 1, 2; k = 1, 2, \dots)$$

where

$$F_{10}(t; u) = - \int_0^1 u^3(x,t) dx,$$

$$\phi_{10} = \int_0^1 \phi(x) dx, \psi_{10} = \int_0^1 \psi(x) dx,$$

$$F_{1k}(t; u) = -2 \int_0^1 u^3(x,t) \cos \lambda_k x dx,$$

$$\phi_{1k} = 2 \int_0^1 \phi(x) \cos \lambda_k x dx, \psi_{1k} = 2 \int_0^1 \psi(x) \cos \lambda_k x dx,$$

$$F_{2k}(t; u) = -2 \int_0^1 u^3(x,t) \sin \lambda_k x dx,$$

$$\phi_{2k} = 2 \int_0^1 \phi(x) \sin \lambda_k x dx, \psi_{2k} = 2 \int_0^1 \psi(x) \sin \lambda_k x dx.$$

It is clear that  $\lambda_k^4 - \beta \lambda_k^2 + \alpha = \left( \lambda_k^2 - \frac{\beta}{2} \right)^2 + \alpha - \frac{\beta^2}{4}$ .

Let suppose that,  $\beta^2 < 4\alpha$ . Then, by solving problem (5)-(8) we find:

$$u_{10}(t) = \frac{1}{\beta_0 \rho_0(T)} \left\{ \beta_0 (\cos \beta_0 t + \delta \cos \beta_0 (T-t)) \phi_{10} \right.$$

$$+ (\sin \beta_0 t - \delta \sin \beta_0 (T-t)) \psi_{10}$$

$$\left. - \delta \int_0^T F_{10}(\tau; u) (\sin \beta_0 (T+t-\tau) + \delta \sin \beta_0 (t-\tau)) d\tau \right\} \tag{9}$$

$$+ \frac{1}{\beta_0} \int_0^t F_{10}(\tau; u) \sin \beta_0 (t-\tau) d\tau,$$

$$u_{ik}(t) = \frac{1}{\beta_k \rho_k(T)} \left\{ \beta_k (\cos \beta_k t + \delta \cos \beta_k (T-t)) \phi_{ik} \right.$$

$$+ (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{ik}$$

$$\left. - \delta \int_0^T F_{ik}(\tau; u) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\} \tag{10}$$

$$+ \frac{1}{\beta_k} \int_0^t F_{ik}(\tau; u) \sin \beta_k (t-\tau) d\tau \quad (i = 1, 2; k = 1, 2, \dots),$$

where

$$\beta_k = \sqrt{\lambda_k^4 - \beta \lambda_k^2 + \alpha}, \rho_k(T) = 1 + 2\delta \cos \beta_k T + \delta^2,$$

$$k = 0, 1, 2, \dots$$

It is known that

$$u_{ik}(t) = \frac{1}{\beta_k \rho_k(T)} \left\{ \beta_k (\cos \beta_k t + \delta \cos \beta_k (T-t)) \phi_{ik} \right.$$

$$+ (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{ik}$$

$$\left. - \delta \int_0^T F_{ik}(\tau; u) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\}$$

$$+ \frac{1}{\beta_k} \int_0^t F_{ik}(\tau; u) \sin \beta_k (t-\tau) d\tau$$

$$(i = 1, 2; k = 1, 2, \dots),$$

$$u'_{10}(t) = \frac{1}{\rho_0(T)} \left\{ \beta_0 (-\sin \beta_0 t + \delta \sin \beta_0 (T-t)) \phi_0 \right.$$

$$+ (\cos \beta_0 t + \delta \cos \beta_0 (T-t)) \psi_0$$

$$\left. - \delta \int_0^T F_{10}(\tau; u) (\cos \beta_0 (T+t-\tau) + \delta \cos \beta_0 (t-\tau)) d\tau \right\} \tag{11}$$

$$+ \int_0^t F_{10}(\tau; u) \cos \beta_0 (t-\tau) d\tau,$$

$$u'_{ik}(t) = \frac{1}{\rho_k(T)} \left\{ \beta_k (-\sin \beta_k t + \delta \sin \beta_k (T-t)) \phi_{ik} \right.$$

$$+ (\cos \beta_k t + \delta \cos \beta_k (T-t)) \psi_{ik}$$

$$\left. - \delta \int_0^T F_{ik}(\tau; u) (\cos \beta_k (T+t-\tau) + \delta \cos \beta_k (t-\tau)) d\tau \right\} \tag{12}$$

$$+ \int_0^t F_{ik}(\tau; u) \cos \beta_k (t-\tau) d\tau \quad (i = 1, 2; k = 1, 2, \dots),$$

$$u'_{10}(t) = F_{10}(t; u) - \frac{\beta_0}{\rho_0(T)} \left\{ \beta_0 (\cos \beta_0 t \right.$$

$$+ \delta \sin \beta_0 (T-t)) \phi_{10} + (\sin \beta_0 t - \delta \sin \beta_0 (T-t)) \psi_{10}$$

$$\left. - \delta \int_0^T F_{10}(\tau; u) (\sin \beta_0 (T+t-\tau) + \delta \sin \beta_0 (t-\tau)) d\tau \right\} \tag{13}$$

$$- \beta_0 \int_0^t F_{10}(\tau; u) \sin \beta_0 (t-\tau) d\tau.$$

$$\begin{aligned}
 u_{ik}''(t) = & F_{ik}(t;u) - \frac{\beta_k}{\rho_k(T)} \{ \beta_k (\cos \beta_k t \\
 & + \delta \sin \beta_k (T-t)) \phi_{ik} + (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{ik} \\
 & - \delta \int_0^T F_{ik}(\tau;u) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \} \quad (14) \\
 & - \beta_k \int_0^t F_{ik}(\tau;u) \sin \beta_k (t-\tau) d\tau \quad (i=1,2; k=1,2,\dots).
 \end{aligned}$$

After substituting the expressions  $u_{1k}(t)$  ( $k=0,1,2,\dots$ ) and  $u_{2k}(t)$  ( $k=1,2,\dots$ ) into (4), we get:

$$\begin{aligned}
 u(x,t) = & \frac{1}{\beta_0 \rho_0(T)} \{ \beta_0 (\cos \beta_0 t + \delta \cos \beta_0 (T-t)) \phi_{10} \\
 & + (\sin \beta_0 t - \delta \sin \beta_0 (T-t)) \psi_{10} \\
 & - \delta \int_0^T F_{10}(\tau;u) (\sin \beta_0 (T+t-\tau) + \delta \sin \beta_0 (t-\tau)) d\tau \} \\
 & + \frac{1}{\beta_0} \int_0^t F_{10}(\tau;u) \sin \beta_0 (t-\tau) d\tau \\
 & + \sum_{k=1}^{\infty} \left\{ \frac{1}{\beta_k \rho_k(T)} [ \beta_k (\cos \beta_k t + \delta \cos \beta_k (T-t)) \phi_{1k} \right. \\
 & + (\sin \beta_k t + \delta \sin \beta_k (T-t)) \psi_{1k} \\
 & \left. - \delta \int_0^T F_{1k}(\tau;u) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\} \\
 & + \frac{1}{\beta_k} \int_0^t F_{1k}(\tau;u) \sin \beta_k (t-\tau) d\tau \} \cos \lambda_k x \\
 & + \sum_{k=1}^{\infty} \left\{ \frac{1}{\beta_k \rho_k(T)} [ \beta_k (\cos \beta_k t + \delta \cos \beta_k (T-t)) \phi_{2k} \right. \\
 & + (\sin \beta_k t + \delta \sin \beta_k (T-t)) \psi_{2k} \\
 & \left. - \delta \int_0^T F_{2k}(\tau;u) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \right\} \quad (15) \\
 & + \frac{1}{\beta_k} \int_0^t F_{2k}(\tau;u) \sin \beta_k (t-\tau) d\tau \} \sin \lambda_k x.
 \end{aligned}$$

Thus, the solution of problems (1)-(3) is reduced to the solution of integral equation (15) with respect to the unknown function  $u(x,t)$ .

Similarly to [12], it is possible to prove the following lemma.

**Lemma**

If  $u(x,t)$  is any classical solution of problem (1)-(3), the functions

$$\begin{aligned}
 u_{10}(t) &= \int_0^1 u(x,t) dx, \\
 u_{1k}(t) &= 2 \int_0^1 u(x,t) \cos \lambda_k x dx,
 \end{aligned}$$

$$u_{2k}(t) = 2 \int_0^1 u(x,t) \sin \lambda_k x dx \quad (k=1,2,\dots).$$

satisfy the systems (9), (10) in  $[0,T]$ .

From the lemma indicated above it follows that if

$$\begin{aligned}
 u_{10}(t) &= \int_0^1 u(x,t) dx, \quad u_{1k}(t) = 2 \int_0^1 u(x,t) \cos \lambda_k x dx, \\
 u_{2k}(t) &= 2 \int_0^1 u(x,t) \sin \lambda_k x dx \quad (k=1,2,\dots)
 \end{aligned}$$

is the solution of systems (9), (10) then the function

$$\begin{aligned}
 u(x,t) = & \sum_{k=0}^{\infty} u_{1k}(t) \cos \lambda_k x \\
 & + \sum_{k=1}^{\infty} u_{2k}(t) \sin \lambda_k x \quad (\lambda_k = 2k\pi)
 \end{aligned}$$

is the solution of (15).

From the above mentioned lemma follows

**Corollary**

Suppose that equation (15) has a unique solution. Then the problem (1)-(3) may have at most one solution, i.e. of the solution of problem (1)-(3) exists it is unique.

**3. Existence and Uniqueness of the Classical Solution**

Denote by  $B_{2,T}^5$  [13] the set of all functions  $u(x,t)$  of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} u_{2k}(t) \sin \lambda_k x \quad (\lambda_k = 2k\pi),$$

defined on  $D_T$ , where each of the functions  $u_{1k}(t)$  ( $k=0,1,\dots$ ),  $u_{2k}(t)$  ( $k=1,2,\dots$ ) are continuous on  $[0,T]$  and

$$\begin{aligned}
 J(u) = & \|u_{10}(t)\|_{C[0,T]} + \left( \sum_{k=1}^{\infty} (\lambda_k^5 \|u_{1k}(t)\|_{C[0,T]})^2 \right)^{1/2} \\
 & + \left( \sum_{k=1}^{\infty} (\lambda_k^5 \|u_{2k}(t)\|_{C[0,T]})^2 \right)^{1/2} < +\infty.
 \end{aligned}$$

We define the norm in this set as follows:

$$\|u(x,t)\|_{B_{2,T}^5} = J(u).$$

It is known that  $B_{2,T}^5$  is a Banach space.

Now in the space  $B_{2,T}^5$  we consider the operator

$$\Phi(u) = \sum_{k=0}^{\infty} \Phi_{1k}(t) \cos \lambda_k x + \sum_{k=1}^{\infty} \Phi_{2k}(t) \sin \lambda_k x,$$

where

$$\begin{aligned} \Phi_{10}(t) &= \frac{1}{\beta_0 \rho_0(T)} \{ \beta_0 (\cos \beta_0 t + \delta \cos \beta_0 (T-t)) \phi_{10} \\ &+ (\sin \beta_0 t - \delta \sin \beta_0 (T-t)) \psi_{10} \\ &- \delta \int_0^T F_{10}(\tau; u) (\sin \beta_0 (T+t-\tau) + \delta \sin \beta_0 (t-\tau)) d\tau \} \\ &+ \frac{1}{\beta_0} \int_0^t F_{10}(\tau; u) \sin \beta_0 (t-\tau) d\tau, \\ \Phi_{ik}(t) &= \frac{1}{\beta_k \rho_k(T)} \{ \beta_k (\cos \beta_k t + \delta \cos \beta_k (T-t)) \phi_{ik} \\ &+ (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{ik} \\ &- \delta \int_0^T F_{ik}(\tau; u) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \} \\ &+ \frac{1}{\beta_k} \int_0^t F_{ik}(\tau; u) \sin \beta_k (t-\tau) d\tau \quad (i = 1, 2; k = 1, 2, \dots) \end{aligned}$$

Hence, we get:

$$\begin{aligned} &\|\Phi_{10}(t)\|_{C[0,T]} \\ &\leq \frac{1}{\beta_0 |\rho_0(T)|} \{ \beta_0 (1+|\delta|) |\phi_{10}| + (1+|\delta|) |\psi_{10}| \} \\ &+ \frac{1}{\beta_0} \left( 1 + \frac{1}{\rho_0(T)} |\delta| (1+|\delta|) \sqrt{T} \left( \int_0^T |F_{10}(\tau; u)|^2 d\tau \right)^{\frac{1}{2}} \right. \\ &\quad \left. \left( \sum_{k=1}^{\infty} (\lambda_k^5 \|\Phi_{ik}(t)\|_{C[0,T]})^2 \right)^{1/2} \right. \\ &\leq \sqrt{3} \rho(T) (1+|\delta|) \left( \sum_{k=1}^{\infty} (\lambda_k^5 |\phi_{ik}|)^2 \right)^{1/2} \\ &+ \sqrt{3} \rho(T) (1+|\delta|) \varepsilon \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\psi_{ik}|)^2 \right)^{1/2} \\ &+ \sqrt{3T} (1+|\delta|) \rho(T) (1+|\delta|) \\ &\quad \left. \varepsilon \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |F_{ik}(\tau; u)|)^2 d\tau \right)^{1/2} \right. \end{aligned} \tag{16}$$

where

$$\rho_k(T) \equiv \sup_k \rho_k^{-1}(T) \leq 1 / (1 + \delta^2 - 2|\delta|),$$

$$\sup_k \left( \frac{\lambda_k^2}{\sqrt{\lambda_k^4 - \beta \lambda_k^2 + \alpha}} \right) = \frac{1}{\varepsilon}.$$

Assume that the data of problem (1)-(3) satisfy the following conditions:

1.  $\phi(x) \in C^4[0,1], \quad \phi^{(5)}(x) \in L_2(0,1), \quad \phi(0) = \phi(1),$   
 $\phi'(0) = \phi'(1), \quad \phi''(0) = \phi''(1), \quad \phi^{(3)}(0) = \phi^{(3)}(1),$   
 $\phi^{(4)}(0) = \phi^{(4)}(1);$

2.  $\psi(x) \in C^2[0,1], \quad \psi'''(x) \in L_2(0,1), \quad \psi(0) = \psi(1),$   
 $\psi'(0) = \psi'(1), \quad \psi''(0) = \psi''(1).$

Then from (12) we have

$$\begin{aligned} \|\Phi_{10}(t)\|_{C[0,T]} &\leq \frac{1}{\rho_0(T)} (1+|\delta|) \|\phi(x)\|_{L_2(0,1)} \\ &+ \frac{1}{\beta_0 \rho_0(T)} (1+|\delta|) \|\psi(x)\|_{L_2(0,1)} \\ &+ \frac{1}{\beta_0} \left( 1 + \frac{1}{\rho_0(T)} |\delta| (1+|\delta|) \right) \sqrt{T} \|u^3(x,t)\|_{L_2(D_T)}, \\ &\left( \sum_{k=1}^{\infty} (\lambda_k^5 \|\Phi_k(t)\|_{C[0,T]})^2 \right)^{1/2} \\ &\leq \sqrt{3} \rho(T) (1+|\delta|) \|\phi^{(5)}(x)\|_{L_2(0,1)} \\ &+ \sqrt{3} \rho(T) (1+|\delta|) \varepsilon \|\psi'''(x)\|_{L_2(0,1)} \\ &+ \sqrt{3T} (1+|\delta|) \rho(T) (1+|\delta|) \varepsilon \\ &\quad \times \|6u_x^3 + 18u \cdot u_x \cdot u_{xx} + 3u^2 \cdot u_{xxx}\|_{L_2(D_T)}. \end{aligned} \tag{17}$$

Denote

$$\begin{aligned} A(T) &= 6\rho(T)(1+|\delta|) \|\phi^{(5)}(x)\|_{L_2(0,1)} \\ &+ 6\rho(T)(1+|\delta|) \varepsilon \|\psi'''(x)\|_{L_2(0,1)} \\ &+ \frac{1}{\rho_0(T)} (1+|\delta|) \|\phi(x)\|_{L_2(0,1)} \\ &+ \frac{1}{\beta_0} \left( 1 + \frac{1}{\rho_0(T)} |\delta| (1+|\delta|) \right) \|\psi(x)\|_{L_2(0,1)}, \\ B(T) &= \sqrt{3T} (1+|\delta|) \rho(T) (1+|\delta|) \varepsilon \\ &+ \frac{1}{\beta_0} \left( 1 + \frac{1}{\rho_0(T)} |\delta| (1+|\delta|) \right) \sqrt{T}. \end{aligned}$$

and rewrite (17) in the form:

$$\begin{aligned} \|\Phi u\|_{B_{2,T}^5} &\leq A(T) + B(T) \\ &\quad \times \left( \|6u_x^3 + 18u \cdot u_x \cdot u_{xx} + 3u^2 \cdot u_{xxx}\|_{L_2(D_T)} \right. \\ &\quad \left. + \|u^3(x,t)\|_{L_2(D_T)} \right). \end{aligned} \tag{18}$$

**Theorem**

Let conditions 1-2,  $\delta \neq \pm 1, \beta < 4\alpha$  be fulfilled. Then for sufficiently small values of  $T$ , problem (1)-(3) has a unique classical solution in the ball  $K = K_R (\|u\|_{B_{2,T}^5} \leq A(T) + 1)$  of the space  $B_{2,T}^5$ .

**Proof:**

In the space  $B_{2,T}^5$  consider the equation

$$u = \Phi u, \tag{19}$$

where the operator  $\Phi$  is defined by the right hand side of equation (15). Consider the operator in the ball  $K = K_R (\|u\|_{B_{2,T}^5} \leq R = A(T) + 1)$  from  $B_{2,T}^5$ .

Similarly, from (18) we get that for any  $u, u_1, u_2 \in K_R$ , the following estimates are valid

$$\|\Phi u\|_{B_{2,T}^5} \leq A(T) + 28B(T)\sqrt{TR}^3, \tag{20}$$

$$\|\Phi u_1 - \Phi u_2\|_{B_{2,T}^5} \leq 82R^2\sqrt{T} \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^3}. \tag{21}$$

It follows from the estimates (20), (21) that for rather small values of  $T$  the operator  $\Phi$  acts in the ball  $K = K_R$  and is contractive. Therefore in the ball  $K = K_R$  it has unique fixed point  $\{u\}$ , that is a unique solution of equation (19). Moreover, integral equation (15) also has a unique solution belonging to the ball  $K = K_R$ .

The function  $u(x,t)$ , as an element of the space  $B_{2,T}^5$ , is continuous and has continuous derivatives  $u_x(x,t)$ ,  $u_{xx}(x,t)$ ,  $u_{xxx}(x,t)$ ,  $u_{xxxx}(x,t)$  on  $D_T$ .

Now we'll show that  $u_{tt}(x,t)$  is continuous in  $D_T$ . Allowing for (10), from (6) we have

$$\begin{aligned} u''_{ik}(t) = & \frac{1}{\beta_k \rho_k(T)} \{ \beta_k (\cos \beta_k t + \delta \cos \beta_k (T-t)) \phi_{ik} \\ & + (\sin \beta_k t - \delta \sin \beta_k (T-t)) \psi_{ik} \\ & - \delta \int_0^T F_{ik}(\tau; u) (\sin \beta_k (T+t-\tau) + \delta \sin \beta_k (t-\tau)) d\tau \} \\ & + \frac{1}{\beta_k} \int_0^t F_{ik}(\tau; u) \sin \beta_k (t-\tau) d\tau + F_{ik}(t; u) \end{aligned}$$

$(i = 1, 2; k = 1, 2, \dots)$ .

Hence, we have:

$$\begin{aligned} & \left( \sum_{k=1}^{\infty} (\lambda_k \|u''_{ik}(t)\|_{C[0,T]})^2 \right)^{1/2} \\ & \leq 2(1 + \beta + \alpha) \left[ \rho(T)(1 + |\delta|) \left( \sum_{k=1}^{\infty} (\lambda_k^5 |\phi_{ik}|)^2 \right)^{1/2} \right. \\ & \quad + \varepsilon \rho(T)(1 + |\delta|) \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\psi_{ik}|)^2 \right)^{1/2} \\ & \quad \left. + \sqrt{T}(1 + |\delta|) \rho(T)(1 + |\delta|) \varepsilon \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k^3 |F_{ik}(\tau; u)|)^2 d\tau \right)^{1/2} \right] \\ & \quad + 2 \left( \sum_{k=1}^{\infty} (\lambda_k \|F_{ik}(t; u)\|_{C[0,T]})^2 \right)^{1/2}, \quad (i = 1, 2), \end{aligned}$$

or

$$\begin{aligned} & \left( \sum_{k=1}^{\infty} (\lambda_k \|u''_{ik}(t)\|_{C[0,T]})^2 \right)^{1/2} \\ & \leq 2(1 + \beta + \alpha) \left( \rho(T)(1 + |\delta|) \|\phi^{(5)}(x)\|_{L_2(0,1)} \right. \\ & \quad + \rho(T)(1 + |\delta|) \varepsilon \|\psi'''(x)\|_{L_2(0,1)} \\ & \quad \left. + T(1 + |\delta|) \rho(T)(1 + |\delta|) \varepsilon \right) \end{aligned}$$

$$\begin{aligned} & \times \left\| 6u_x^3 + 18u \cdot u_x \cdot u_{xx} + 3u^2 \cdot u_{xxx} \right\|_{L_2(D_T)} \\ & + 2 \left\| 3u^2 \cdot u_x \right\|_{C[0,T]} \Big\|_{L_2(0,1)} \quad (i = 1, 2). \end{aligned}$$

It follows from the last relation that the function  $u_{tt}(x,t)$  is continuous in  $D_T$ .

It is easy to verify that equation (1) and conditions (2), (3) are satisfied in the ordinary sense. So,  $u(x,t)$  is the solution of the problem (1)-(3) in the ball  $K = K_R$  from  $B_{2,T}^5$ . Since equation (15) has a unique solution in the ball  $K = K_R$  from  $B_{2,T}^5$ . By the above mentioned corollary the problem (1)-(3) has a unique classical solution in the ball  $K = K_R$  from  $B_{2,T}^5$ . The theorem is thus proved.

### 4. Conclusion

The following results have been obtained:

1. The existence of the solution of a nonlocal boundary value problem for the equation of motion of a homogeneous bar is proved;
2. The uniqueness of the solution of a nonlocal boundary value problem for the equation of motion of a homogeneous bar is shown.

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