

Derivation of Continuous Linear Multistep Methods Using Hermite Polynomials as Basis Functions

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Abstract This paper concerns the derivation of continuous linear multistep methods for solving first-order initial value problems (IVPs) of ordinary differential equations (ODEs) with step number $k = 3$ using Hermite polynomials as basis functions. Adams-Bashforth, Adams-Moulton and optimal order methods are derived through collocation and interpolation technique. The derived methods are applied to solve two first order initial value problems of ordinary differential equations. The result obtained by the optimal order method compared favourably with those of the standard existing methods of Adams-Bashforth and Adams-Moulton.

Keywords: linear multistep method, hermite polynomial, collocation, interpolation, optimal order scheme, ordinary differential equation, initial value problem

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1. Introduction

Linear multistep methods (LMMs) are very popular for solving initial value problems (IVPs) of ordinary differential equations (ODEs). They are also applied to solve higher order ODEs. LMMs are not self-starting hence, need starting values from single-step methods like Euler's method and Runge-Kutta family of methods.

The general k -step LMM is as given in Lambert (1973)

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j} \quad (1)$$

where α_j and β_j are uniquely determined and $\alpha_0 + \beta_0 \neq 0$, $\alpha_k = 1$.

The LMM in Equation (1) generates discrete schemes which are used to solve first-order ODEs. Other researchers have introduced the continuous LMM using the continuous collocation and interpolation approach leading to the development of the continuous LMMs of the form

$$y(x) = \sum_{j=0}^k \alpha_j(x) y_{n+j} + h \sum_{j=0}^k \beta_j(x) f_{n+j} \quad (2)$$

where α_j and β_j are expressed as continuous functions of x and are at least differentiable once [8].

According to Okunuga and Ehigie [17], the existing methods of deriving the LMMs in discrete form include the interpolation approach, numerical integration, Taylor series expansion and through the determination of the order of LMM. Continuous collocation and interpolation

technique is now widely used for the derivation of LMMs, block methods and hybrid methods.

Several continuous LMMs have been derived using different techniques and approaches: Alabi [5] derived continuous solvers of IVPs using Chebyshev polynomial in a multistep collocation technique; Okunuga and Ehigie [17] derived two-step continuous and discrete LMMs using power series as basis function; Mohammed [15] derived a linear multistep method with continuous coefficients and used it to obtain multiple finite difference methods which were directly applied to solve first-order ODEs; Odekunle *et al* [16] developed a continuous linear multistep method using interpolation and collocation for the solution of first-order ODE with constant stepsize; Adesanya *et al* [2,3] considered the method of collocation of the differential system and interpolation of the approximate solution to generate a continuous LMM, which is solved for the independent solution to yield a continuous block method; James *et al* [11,12] proposed a continuous block method for the solution of second order IVPs with constant stepsize, the method was developed by interpolation and collocation of power series approximate solution; Anake [6] developed a new class of continuous implicit hybrid one-step methods capable of solving IVPs of general second order ODEs using the collocation and interpolation techniques of the power series approximate solution; James *et al* [11,12] adopted the method of collocation and interpolation of power series approximate solution to generate a continuous LMM; Ehigie *et al* [9] proposed a two-step continuous multistep method of hybrid type for the direct integration of second order ODEs in a multistep collocation technique; Akinfenwa *et al* [4] developed a four step continuous block hybrid method with four non-step points for the direct solution of first-order IVPs; Adesanya *et al* [2,3] adopted the method

of collocation of the differential system and interpolation of the approximate solution at grid and off grid points to yield a continuous LMM with constant stepsize and James *et al* [11,12] developed a continuous block method using the approach of collocation of the differential system and interpolation of the power series approximate solution.

The introduction of continuous collocation schemes is of great importance as better global error can be estimated and approximations can be equally obtained. Also, the gap between the discrete collocation methods and the conventional multistep methods is bridged. In this study, we will develop continuous multistep collocation methods for the solution of first-order IVPs of ODEs using the probabilists' Hermite polynomials as the basis function. The corresponding discrete schemes shall also be obtained.

2. Methods

In Awoyemi [7] and Onumanyi *et al* [18], some continuous LMM of the type in Equation (2) were developed using the collocation function of the form:

$$y(x) = \sum_{j=0}^k \alpha_j x^j. \tag{3}$$

Awoyemi *et al* [8] proposed a similar function of the type in Equation (3)

$$y(x) = \sum_{j=0}^k \alpha_j (x - x_k)^j \tag{4}$$

to develop LMM for the solution of third-order IVPs. Adeniyi and Alabi [1] used Chebyshev polynomial function of the form:

$$y(x) = \sum_{j=0}^M \alpha_j T_j \left(\frac{x - x_k}{h} \right),$$

where $T_j(x)$ are some Chebyshev function to develop continuous LMM.

In this paper, we propose the Probabilists' Hermite polynomial of the form [13]:

$$y(x) = \sum_{j=0}^k a_j H_j(x - x_k)$$

where $H_j(x)$ are probabilists' Hermite polynomials generated by the formula:

$$H_n(x) = (-1)^n e^{\left(\frac{x^2}{2}\right)} \frac{d^n}{dx^n} e^{-\left(\frac{x^2}{2}\right)} = \left(x - \frac{d}{dx}\right)^n \cdot 1$$

and whose recursive relation is

$$H_{n+1}(x) = xH_n(x) - H_n'(x)$$

to develop continuous LMMs for the solution of first-order IVPs of ODEs of the form:

$$y' = f(x, y(x)), y(x_0) = y_0. \tag{5}$$

The first six probabilists' Hermite polynomials are:

$$H_0 = 1, H_1 = x, H_2 = x^2 - 1,$$

$$H_3 = x^3 - 3x, H_4 = x^4 - 6x^2 + 3, H_5 = x^5 - 10x^3 + 15x.$$

2.1. Derivation of the Linear Multistep Methods

We wish to approximate the exact solution $y(x)$ to the IVP in Equation (5) by a polynomial of degree n of the form:

$$y(x) = \sum_{j=0}^n a_j H_j(x - x_k), \quad x_k \leq x \leq x_{k+p} \tag{6}$$

which satisfies the equations

$$\left. \begin{aligned} y'(x) &= f(x, y(x)), x_k \leq x \leq x_{k+p} \\ y(x_k) &= y_k. \end{aligned} \right\} \tag{7}$$

The Adams-Bashforth Method

To derive the three-step Adams-Bashforth method, we set $n = 3$ in Equation (6), yielding

$$y(x) = a_0 + a_1(x - x_k) + a_2 \left[(x - x_k)^2 - 1 \right] + a_3 \left[(x - x_k)^3 - 3(x - x_k) \right]. \tag{8}$$

Differentiating once gives the equation:

$$y'(x) = a_1 + 2a_2(x - x_k) + 3a_3 \left[(x - x_k)^2 - 1 \right]. \tag{9}$$

Interpolating Equation (8) at $x = x_{k+2}$ and collocating Equation (9) at $x = x_k, x_{k+1}, x_{k+2}$ yields

$$y(x_{k+2}) = a_0 + a_1(x_{k+2} - x_k) + a_2 \left[(x_{k+2} - x_k)^2 - 1 \right] + a_3 \left[(x_{k+2} - x_k)^3 - 3(x_{k+2} - x_k) \right] = y_{k+2},$$

$$y'(x_k) = a_1 - 3a_3 = f_k,$$

$$y'(x_{k+1}) = a_1 + 2a_2 \left(\begin{smallmatrix} x_{k+1} \\ -x_k \end{smallmatrix} \right) + 3a_3 \left[(x_{k+1} - x_k)^2 - 1 \right] = f_{k+1},$$

$$y'(x_{k+2}) = a_1 + 2a_2 \left(\begin{smallmatrix} x_{k+2} \\ -x_k \end{smallmatrix} \right) + 3a_3 \left[(x_{k+2} - x_k)^2 - 1 \right] = f_{k+2}.$$

In matrix form, we have:

$$\begin{pmatrix} 1 & 2h & (4h^2 - 1) & (8h^3 - 6h) \\ 0 & h & 0 & -3h \\ 0 & h & 2h^2 & (3h^3 - 3h) \\ 0 & h & 4h^2 & (12h^3 - 3h) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_{k+2} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \end{pmatrix}.$$

Solving the system of equations by Gaussian elimination we obtain:

$$a_0 = y_{k+2} - \frac{h}{3}(f_{k+2} + 4f_{k+1} + f_k) + \frac{1}{4h}(-f_{k+2} + 4f_{k+1} - 3f_k),$$

$$a_1 = f_k + \frac{1}{2h^2}(f_{k+2} - 2f_{k+1} + f_k),$$

$$a_2 = \frac{1}{4h}(-f_{k+2} + 4f_{k+1} - 3f_k),$$

$$a_3 = \frac{1}{6h^2}(f_{k+2} - 2f_{k+1} + f_k).$$

Substituting for $a_j, j = 0, 1, 2, 3$ in Equation (8) yields the continuous method:

$$y(x) = y_{k+2} - \frac{h}{3}(f_{k+2} + 4f_{k+1} + f_k) + f_k(x - x_k) + \frac{1}{4h}(-f_{k+2} + 4f_{k+1} - 3f_k)(x - x_k)^2 + \frac{1}{6h^2}(f_{k+2} - 2f_{k+1} + f_k)(x - x_k)^3. \tag{10}$$

Evaluating Equation (10) at $x = x_{k+3}$, we obtain the discrete form as:

$$y_{k+3} = y_{k+2} + \frac{h}{12}(23f_{k+2} - 16f_{k+1} + 5f_k). \tag{11}$$

The Adams-Moulton Method

To derive the three-step Adams-Moulton method, we let $n = 4$ in Equation (6) and differentiate the obtained function once. This gives:

$$y(x) = a_0 + a_1(x - x_k) + a_2[(x - x_k)^2 - 1] + a_3[(x - x_k)^3 - 3(x - x_k)] + a_4[(x - x_k)^4 - 6(x - x_k)^2 + 3], \tag{12}$$

$$y'(x) = a_1 + 2a_2(x - x_k) + 3a_3[(x - x_k)^2 - 1] + a_4[4(x - x_k)^3 - 12(x - x_k)]. \tag{13}$$

Interpolating Equation (12) at $x = x_{k+2}$ and collocating Equation (13) at $x = x_k, x_{k+1}, x_{k+2}, x_{k+3}$ gives to the equations:

$$\left. \begin{aligned} y(x_{k+2}) &= a_0 + a_1(x_{k+2} - x_k) + a_2[(x_{k+2} - x_k)^2 - 1] + a_3[(x_{k+2} - x_k)^3 - 3(x_{k+2} - x_k)] + a_4[(x_{k+2} - x_k)^4 - 6(x_{k+2} - x_k)^2 + 3] = y_{k+2}, \\ y'(x_k) &= a_1 - 3a_3 = f_k, \\ y'(x_{k+1}) &= a_1 + 2a_2(x_{k+1} - x_k) + 3a_3[(x_{k+1} - x_k)^2 - 1] + a_4[4(x_{k+1} - x_k)^3 - 12(x_{k+1} - x_k)] = f_{k+1}, \\ y'(x_{k+2}) &= a_1 + 2a_2(x_{k+2} - x_k) + 3a_3[(x_{k+2} - x_k)^2 - 1] + a_4[4(x_{k+2} - x_k)^3 - 12(x_{k+2} - x_k)] = f_{k+2}, \\ y'(x_{k+3}) &= a_1 + 2a_2(x_{k+3} - x_k) + 3a_3[(x_{k+3} - x_k)^2 - 1] + a_4[4(x_{k+3} - x_k)^3 - 12(x_{k+3} - x_k)] = f_{k+3}. \end{aligned} \right\} \tag{14}$$

The equations can be written in matrix form:

$$\begin{pmatrix} 1 & 2h & (4h^2 - 1) & (8h^3 - 6h) & (16h^4 - 24h^2 + 3) \\ 0 & h & 0 & -3h & 0 \\ 0 & h & 2h^2 & (3h^3 - 3h) & (4h^4 - 12h^2) \\ 0 & h & 4h^2 & (12h^3 - 3h) & (32h^4 - 24h^2) \\ 0 & h & 6h^2 & (27h^3 - 3h) & (108h^4 - 36h^2) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_{k+2} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \\ hf_{k+3} \end{pmatrix}.$$

Solving the system of equations by Gaussian elimination, we have that:

$$\left. \begin{aligned} a_0 &= y_{k+2} + \frac{h}{12}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k) + \frac{1}{8h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k) - \frac{h}{3}(f_{k+2} + 4f_{k+1} + f_k), \\ a_1 &= f_k + \frac{1}{2h^2}(-f_{k+3} + 4f_{k+2} - 5f_{k+1} + 2f_k), \\ a_2 &= \frac{1}{12h}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k) + \frac{1}{4h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k), \\ a_3 &= \frac{1}{6h^2}(-f_{k+3} + 4f_{k+2} - 5f_{k+1} + 2f_k), \\ a_4 &= \frac{1}{24h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k). \end{aligned} \right\} \tag{15}$$

Substituting for $a_j, j = 0, 1, 2, 3, 4$ in Equation (12) yields the continuous method

$$y(x) = y_{k+2} - \frac{h}{3}(f_{k+2} + 4f_{k+1} + f_k) + f_k(x - x_k) + \frac{1}{12h}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k)(x - x_k)^2 + \frac{1}{6h^2}(-f_{k+3} + 4f_{k+2} - 5f_{k+1} + 2f_k)(x - x_k)^3 + \frac{1}{24h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k)(x - x_k)^4. \tag{16}$$

Evaluating Equation (16) at $x = x_{k+3}$, we obtain the discrete scheme:

$$y_{k+3} = y_{k+2} + \frac{h}{24}(9f_{k+3} + 19f_{k+2} - 5f_{k+1} + f_k). \tag{17}$$

The Optimal Order Method

The optimal order scheme is an implicit multistep method similar to the Adams-Moulton method. To derive the three-step optimal order method, we shall consider the system of equations in Equation (14) except for $y(x_{k+2})$. Interpolating Equation (12) at x_{k+1} we have

$$y(x_{k+1}) = a_0 + a_1(x_{k+1} - x_k) + a_2[(x_{k+1} - x_k)^2 - 1] + a_3[(x_{k+1} - x_k)^3 - (x_{k+1} - x_k)] + a_4[(x_{k+1} - x_k)^4 - 6(x_{k+1} - x_k)^2 + 3] = y_{k+1}.$$

The corresponding matrix of the equations is:

$$\begin{pmatrix} 1 & h & (h^2-1) & (h^3-3h) & (h^4-6h^2+3) \\ 0 & h & 0 & -3h & 0 \\ 0 & h & 2h^2 & (3h^3-3h) & (4h^4-12h^2) \\ 0 & h & 4h^2 & (12h^3-3h) & (32h^4-24h^2) \\ 0 & h & 6h^2 & (27h^3-3h) & (108h^4-36h^2) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_{k+1} \\ hf_k \\ hf_{k+1} \\ hf_{k+2} \\ hf_{k+3} \end{pmatrix}$$

Solving the system of equations give the same result as in Equation (15) above except for a_0 . Thus, we have a_0 to be:

$$a_0 = y_{k+1} + \frac{1}{12h}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k) + \frac{1}{8h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k) - \frac{h}{24}(f_{k+3} - 5f_{k+2} + 19f_{k+1} + 9f_k).$$

Substituting for $a_j, j = 0, 1, 2, 3, 4$ in Equation (12) yields the continuous three-step optimal order method:

$$y(x) = y_{k+1} - \frac{h}{24} \left(\begin{matrix} f_{k+3} - 5f_{k+2} \\ +19f_{k+1} + 9f_k \end{matrix} \right) + f_k(x - x_k) + \frac{1}{12h}(2f_{k+3} - 9f_{k+2} + 18f_{k+1} - 11f_k)(x - x_k)^2 + \frac{1}{6h^2}(-f_{k+3} + 4f_{k+2} - 5f_{k+1} + 2f_k)(x - x_k)^3 + \frac{1}{24h^3}(f_{k+3} - 3f_{k+2} + 3f_{k+1} - f_k)(x - x_k)^4. \tag{18}$$

Evaluating Equation (18) at $x = x_{k+3}$, we obtain the discrete form:

$$y_{k+3} = y_{k+1} + \frac{h}{3}(f_{k+3} + 4f_{k+2} + f_{k+1}). \tag{19}$$

3. Numerical Examples

In this section, we will apply the derived three-step methods of Adams-Barshforth, Adams-Moulton and the proposed optimal order to solve two IVPs of ODEs. Errors associated with the methods will also be obtained. The four-stage Runge-Kutta method is used to obtain the starting values, and the four-stage Adams-Bashforth method is used as a predictor to the implicit schemes. The result and errors obtained are tabulated for clarity.

Example 1

Consider the IVP

$$y' = -y, 0 \leq x \leq 1, y(0) = 1.$$

Exact Solution: $y(x) = e^{-x}$.

Example 2

Consider the IVP

$$y' = xy, 0 \leq x \leq 1, y(0) = 1.$$

Exact Solution: $y(x) = e^{\left(\frac{x^2}{2}\right)}$.

The error is defined as:

$$Error = |y(x) - y_n(x)|$$

where $y(x)$ is the exact solution and $y_n(x)$ is the approximate solution.

Table 1. Result of Example 1, with stepsize $h = 0.1$

x-value	Exact Solution $y(x)$	Adams-Bashforth Approximation $y_n(x)$	Adams-Moulton Approximation $y_n(x)$	Optimal Order Approximation $y_n(x)$
0.1	0.9048374180	0.9048375000	0.9048375000	0.9048375000
0.2	0.8187307531	0.8187309014	0.8187309014	0.8187309014
0.3	0.7408182207	0.7407858120	0.7408181288	0.7408181824
0.4	0.6703200460	0.6702644224	0.6703196483	0.6703200104
0.5	0.6065306597	0.6064547288	0.6065299217	0.6065303868
0.6	0.5488116361	0.5487200866	0.5488105242	0.5488113400
0.7	0.4965853038	0.4964816829	0.4965838086	0.4965847443
0.8	0.4493289641	0.4492164248	0.4493270829	0.4493283868
0.9	0.4065696597	0.4064508309	0.4065674014	0.4065688105
1.0	0.3678794412	0.3677565415	0.3678768199	0.3678785973

Table 2. Comparison of Absolute Error for Example 1

x-value	Error in Adams-Bashforth Scheme	Error in Adams-Moulton Scheme	Error in Optimal Order Scheme
0.1	8.196404 E-008	8.196404 E-008	8.196404 E-008
0.2	1.483283 E-007	1.483283 E-007	1.483283 E-007
0.3	3.240871 E-005	9.187851 E-007	3.826926 E-008
0.4	5.562367 E-005	3.977782 E-007	3.563031 E-008
0.5	7.593093 E-005	7.379661 E-007	2.728729 E-007
0.6	9.154951 E-005	1.111848 E-006	2.961309 E-007
0.7	1.036209 E-004	1.495200 E-006	5.594595 E-007
0.8	1.125393 E-004	1.881206 E-006	5.773206 E-007
0.9	1.188289 E-004	2.258387 E-006	8.492640 E-007
1.0	1.228997 E-004	2.621222 E-006	8.439141 E-007

Table 3. Result of Example 2, with stepsize $h = 0.1$

x -value	Exact Solution $y(x)$	Adams-Bashforth Approximation $y_n(x)$	Adams-Moulton Approximation $y_n(x)$	Optimal Order Approximation $y_n(x)$
0.1	0.9048374180	1.0050125208	1.0050125208	1.0050125208
0.2	0.8187307531	1.0202013398	1.0202013398	1.0202013398
0.3	0.7408182207	1.0459088908	1.0460285650	1.0460282102
0.4	0.6703200460	1.0830308352	1.0832888045	1.0832874947
0.5	0.6065306597	1.1327285214	1.1331514299	1.1331491504
0.6	0.5488116361	1.1965938880	1.1972215971	1.1972178290
0.7	0.4965853038	1.2767374642	1.2776264969	1.2776213322
0.8	0.4493289641	1.3759040738	1.3771330553	1.3771262162
0.9	0.4065696597	1.4976287157	1.4993061863	1.4992981404
1.0	0.3678794412	1.6464447440	1.6487202423	1.6487114007

Table 4. Comparison of Absolute Error for Example 2

x -value	Error in Adams-Bashforth Scheme	Error in Adams-Moulton Scheme	Error in Optimal Order Scheme
0.1	2.606759 E-010	2.606759 E-010	2.606759 E-010
0.2	2.683873 E-010	2.683873 E-010	2.683873 E-010
0.3	1.189691 E-003	7.050566 E-007	3.503098 E-007
0.4	2.562325 E-003	1.736780 E-006	4.269973 E-007
0.5	4.199317 E-003	2.976807 E-006	6.973629 E-007
0.6	6.234752 E-004	4.233982 E-006	4.659187 E-007
0.7	8.838490 E-004	5.183733 E-006	1.904074 E-009
0.8	1.223690 E-003	5.290980 E-006	1.548156 E-006
0.9	1.673784 E-003	3.686243 E-006	4.359680 E-006
1.0	2.276527 E-003	1.028407 E-006	9.869963 E-006

4. Discussion

Three continuous and discrete LMMs are derived through the technique of collocation and interpolation using the probabilists' Hermite polynomials as basis functions. The result has shown that continuous and discrete LMMs can be derived using any polynomial function and approach. Table 1-Table 4 presents the results by the three derived methods: Equation (11), (17) and (19). Specifically, Table 2 and Table 4 give the errors by these methods. From the results obtained, the Adams-Moulton method produced better results than the Adams-Bashforth method but the proposed optimal order method is the most accurate.

5. Conclusion

The approach and basis functions applied in deriving the LMMs in this paper are different from those of some other researchers, though, the obtained LMMs (continuous and discrete) are same. Also, the proposed optimal order scheme has shown superiority over the standard existing methods of Adams-Bashforth and Adams-Moulton of the same step number, in terms of accuracy.

References

- [1] Adeniyi, R. B. and Alabi, M. O. (2006). Derivation of Continuous Multistep Methods Using Chebyshev polynomial Basis Function. *Abacus*, Vol 33 pp 351-361.
- [2] Adesanya, A. O., Momoh, A. A., Alkali, M. A. and Tahir, A. (2012). Five Steps Block Method for the Solution of Forth Order ODEs. *International Journal of Engineering Research and Applications*, Vol. 2(5) pp 991-998.
- [3] Adesanya, A. O., Odekunle, M. R., Remilekun, M. and James, A. A. (2012). Order Seven Continuous Hybrid Method for the Solution of First Order ODEs. *Canadian Journal on Science and Engineering Mathematics*, Vol. 3(4).
- [4] Akinfewa, O. A., Yao, N. M. and Jator, S. N. (2011). A Linear Multistep Hybrid Method with Continuous Coefficient for Solving Stiff ODE. *Journal of Modern Mathematics and Statistics*, Vol. 5(2) pp 47-53.
- [5] Alabi, M. O. (2008). A Continuous Formulation of Initial Value Solvers with Chebyshev Basis Function in a Multistep Collocation Technique. PhD Thesis. Department of Mathematics, University of Ilorin. 136pp.
- [6] Anake, T. A. (2011). Continuous Implicit Hybrid One-Step Methods for the Solution of Initial Value Problems of General Second Order ODEs. PhD Thesis. School of Postgraduate Studies, Covenant University, Ota, Nigeria. 170pp.
- [7] Awoyemi, D. O (1999). A Class of Continuous Methods for the Solution of General Second Order Ordinary Differential Equations. *International Journal of Computational Mathematics*, Vol 72 pp 29-37.
- [8] Awoyemi, D. O., Kayode, J. S. and Adoghe, L. O. (2014). A Four-Point Fully Implicit Method for the Numerical Integration of Third-Order Ordinary Differential Equations. *International Journal of Physical Sciences*, Vol 9(1) pp 7-12.
- [9] Ehigie, J. O., Okunuga, S. A., Sofoluwe, A. B. and Akanbi, M. A. (2010). On Generalized Two-Step Continuous Linear Multistep Method of Hybrid Type for the Integration of Second Order ODEs. *Archives of Applied Science and Research*, Vol. 2(6) pp 362-372.
- [10] James, A. A., Adesanya, A. O. and Sunday, J. (2013). Continuous Block Method for the Solution of Second Order Initial Value Problems of ODEs. *International Journal of Pure and Applied Mathematics*, Vol. 83(3) pp 405-416.
- [11] James, A. A., Adesanya, A. O., Odekunle, M. R. and Yakubu, D. G. (2013). Constant Order Predictor Corrector Method for the Solution of Medeled Problems of First Order Initial Value Problems of ODEs. *International Journal of Mathematical, Computational, Statistical, Natural and Physical Engineering*, Vol. 7(11).
- [12] James, A. A., Odesanya, A. O., Sunday, J. and Yakubu, D. G. (2013). Half-Step Continuous Block Method for the Solution of Modeled Problems of ODEs. *American Journal of Computational Mathematics*, Vol. 3 pp 261-269.
- [13] Koornwinder, T. H., Wong, R. S. C., Koekoek, R., and Swarttouw, R. F. (2010). Orthogonal Polynomials. In: Olver, F. W. J., Lozier, D. M., Boisvert, R. F. and Clark, C. W. (2010). NIST Handbook

- of Mathematical Functions. Cambridge University Press. ISBN: 978-0521192255.
- [14] Lambert, J. D. (1973). Computational Methods in Ordinary Differential Equations. John Wiley and Sons, New York.
- [15] Mohammed, U. (2011). A Linear Multistep Method with Continuous Coefficients for Solving First-Order ODEs. *Journal of the Nigerian Association of Mathematical Physics*, Vol. 19 pp159-166.
- [16] Odekunle, M. R., Adesanya, A. O. and Sunday, J. (2012). A New Block Integrator for the Solution of Initial Value Problems of First Order ODEs. *International Journal of Pure and Applied Sciences and Technology*, Vol. 11(1) pp92-100.
- [17] Okunuga, S. A. and Ehigie, J. (2009). A New Derivation of Continuous Collocation Multistep Methods Using Power Series as Basis Function. *Journal of Modern Mathematics and Statistics*, Vol 3(2) pp 43-50.
- [18] Onumanyi, P., Oladele, J. O., Adeniyi, R. B. and Awoyemi, D. O. (1993). Derivation of Finite Difference Methods by Collocation. *Abacus*, Vol 23(2) pp 76-83.