American Journal of Applied Mathematics and Statistics, 2014, Vol. 2, No. 5, 330-335
Available online at http://pubs.sciepub.com/ajams/2/5/6

DOI:10.12691/ajams-2-5-6

Recurrence Relations for Single and Product Moments of Generalized Order Statistics from Left Truncated Logistic Distribution

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Received April 25, 2014; Revised August 01, 2014; Accepted September 27, 2014

Abstract In this paper, we establish some recurrence relations satisfied by single and product moments of Generalized Order Statistics from Left Truncated Logistic Distribution. These recurrence relations are independent of left truncated point and therefore are also applicable for Logistic as well as for half Logistic distributions studied in Balakrishnan (1985) and Saran and Pandey (2012). For a particular case these results verify the corresponding results of Saran and Pandey (2004) and Kumar (2010) for p=∞.

Keywords: *order statistics, record values, generalized order statistics, single moment, product moments, recurrence relations, standard logistic distribution, half logistic distribution and truncated distribution*

Cite This Article: Kamal Nain Kapoor, "Recurrence Relations for Single and Product Moments of Generalized Order Statistics from Left Truncated Logistic Distribution." *American Journal of Applied Mathematics and Statistics*, vol. 2, no. 5 (2014): 330-335. doi: 10.12691/ajams-2-5-6.

1. Introduction

Generalized order statistics (GOS) have been introduced and extensively studied in Kamps (1995 a,b) as a unified theoretical set-up which contains a variety of models of ordered random variables with different interpretations. Examples of such models are: Ordinary order statistics, Sequential order statistics, Progressive type II censored order statistics, Record values, k^{th} record value and Pfeifer's records. There is no natural interpretation of generalized order statistics in terms of observed random samples but these models can be effectively applied in life testing and reliability analysis, medical and life time data, and models related to software reliability analysis, etc. The structural similarities and common approach of these models makes it possible to define several distributional properties at once.

2. Left Truncated Logistic Distribution

A random variable X is said to have Left Truncated Logistic Distribution (LTLD) if its probability density function is of the form

$$f(x)=\frac{\left(e^{\beta}+1\right)e^{-x}}{\left(e^{-x}+1\right)^2},\quad \beta<x<\infty,\ \beta>0 \tag{2.1}$$

and its cumulative distribution function is given by

$$F(x)=\frac{e^{\beta}\left(e^{-\beta}-e^{-x}\right)}{e^{-x}+1} \tag{2.2}$$

The characterization differential equation for LTLD distribution is given by

$$\frac{1-F(x)}{f(x)}=1+e^{-x}=\sum_{j=0}^{\infty}\alpha_j x^j, \text{ where}$$

$$\alpha_j=\begin{cases}2, & j=0,\\ \frac{(-1)^j}{j!}, & j\ge 1.\end{cases} \tag{2.3}$$

The mathematical form of pdf, as given in (2.3), is very useful to derive the expressions for recurrence relations for single and product moments of GOS.

3. Generalized Order Statistics

Let $\{X_n, n\ge 1\}$ be a sequence of absolutely continuous, independent and identically distributed random variables with cdf $F(x)=P(X\le x)$ and pdf f(x). Assume k > 0, n ∈ {2,3,…}, $\tilde{m}=(m_1,m_2,...,m_{n-1})\in R^{n-1}$, $M_r=\sum_{j=r}^{n-1}m_j$, such that $\gamma_r=k+n-r+M_r>0$ for all $r\in\{1,2,...,n-1\}$. Then $X(r,n,\tilde{m},k)$, r = 1, 2,…,n, are called GOS if their joint pdf is given by

$$f^{X(1,n,\tilde{m},k),\ X(2,n,\tilde{m},k),\ \ldots,\ X(n,n,\tilde{m},k)}(x_1,x_2,\ldots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}(1-F(x_i))^{m_i} f(x_i)\right)(1-F(x_n))^{k-1} f(x_n), \quad (3.1)$$

where

$$F^{-1}(0+) < x_1 \le x_2 \le \ldots \le x_n < F^{-1}(1).$$

By choosing appropriate values of parameters, we get the distribution of a few very common statistics as shown in the table given below.

S.No.	Choice of parameters for i = 1, 2,…, n	GOS becomes
1	$\gamma_i = n-i+1,\ m_1 = m_2 = \ldots = m_{n-1} = 0$ and k = 1	Joint distribution of n order statistics
2	$\gamma_i = k,\ m_1 = m_2 = \ldots = m_{n-1} = -1,\ k \in N$	k[th] record value
3	$\gamma_i = (n-i+1)\alpha_i\ ,\ \alpha_i > 0$	Sequential order statistics
4	$\gamma_i = \alpha - i + 1,\ \alpha > 0$	Order statistics with non integer sample size
5	$\gamma_i = \beta_i\ ,\ \beta_i > 0$	Pfeifer's record values
6	$m_i \in N_o\ ,\ k \in N$	Progressively type-II right censored order statistics

The joint pdf of first r, GOS is given by:

$$f^{X(1,n,\tilde{m},k),\ X(2,n,\tilde{m},k),\ \ldots,\ X(r,n,\tilde{m},k)}(x_1,x_2,\ldots,x_r) = c_{r-1}\left(\prod_{i=1}^{r-1}(1-F(x_i))^{m_i} f(x_i)\right)(1-F(x_r))^{k+n-r+M_r-1} f(x_r), \quad (3.2)$$

where, $F^{-1}(0+) < x_1 \le x_2 \le \ldots \le x_r < F^{-1}(1)$.

We now consider two cases:

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$

Case II: $\gamma_i \ne \gamma_j$; $i \ne j$, i, j = 1, 2, …, n - 1.

For case I, the GOS will be denoted by X(r, n, m, k). The pdf of X(r, n, m, k) is given by

$$f^{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(1-F(x))^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x)),\quad x \in R \quad (3.3)$$

and the joint pdf of X(r, n, m, k) and X(s, n, m, k), 1 ≤ r < s ≤ n is given by:

$$f^{X(r,n,m,k),\ X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!\,(s-r-1)!}\left((1-F(x))^m f(x)\right) g_m^{r-1}(F(x)).\left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1}(1-F(y))^{\gamma_s-1} f(y)\quad, x<y \quad (3.4)$$

where,

$$c_{r-1} = \prod_{j=1}^{r}\gamma_j,\quad \gamma_j = k+(n-j)(m+1),\ r = 1, 2, \ldots, n,$$

$$g_m(x) = h_m(x) - h_m(0),\quad x \in (0,1),$$

and

$$h_m(x) = \begin{cases} -\dfrac{(1-x)^{m+1}}{m+1}, & m \ne -1, \\ -\log(1-x), & m = -1. \end{cases} \quad (3.5)$$

For case II, the pdf of $X(r,n,\tilde{m},k)$ is given by

$$f^{X(r,n,\tilde{m},k)}(x) = c_{r-1}\, f(x)\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i-1} f(x),\quad x \in R. \quad (3.6)$$

Also, the joint pdf of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, 1 ≤ r < s ≤ n is given by

$$f^{X(r,n,\tilde{m},k),\ X(s,n,\tilde{m},k)}(x,y) = c_{s-1}\left\{\sum_{i=r+1}^{s} a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\} \times\left\{\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right\}\frac{f(x)}{1-F(x)}\frac{f(y)}{1-F(y)}, \quad (3.7)$$

where,

$$c_{s-1} = \prod_{j=1}^{s}\gamma_j,\quad \gamma_j = k+n-j+M_j\ , s = 1, 2, \ldots, n. \quad (3.8)$$

Further it can be proved that

(i) $a_i(r) = \prod_{j(\ne i)=1}^{r}\left(\gamma_j - \gamma_i\right)^{-1}$, $1 \le i \le r \le n$

(ii) $a_i^r(s) = \prod_{j(\ne i)=r+1}^{s}\left(\gamma_j - \gamma_i\right)^{-1}$, $r+1 \le i \le s \le n$.

(iii) $a_i(r) = \left(\gamma_{r+1} - \gamma_i\right) a_i(r+1)$

(iv) $c_r = c_{r-1}\gamma_{r+1}$

(v) $\sum_{i=1}^{r+1} a_i(r+1) = 0$

(vi)

$$\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i} = \frac{(1-F(x))^{\gamma_r}}{(r-1)!} g_m^{r-1}(F(x)) \quad (3.9)$$

(vii)

$$\sum_{i=r+1}^{s} a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} = \frac{(1-F(x))^{-(m+1)(s-r-1)}}{(s-r-1)!}\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s} \times\left(h_m(F(y)) - h_m(F(x))\right)^{s-r-1} \quad (3.10)$$

The moments of order statistics have generated considerable interest in the recent years. The expressions for several recurrence relations and identities satisfied by single as well as product moments of order statistics have been obtained by several authors in the past. These relations help in reducing the quantum of computations involved. Joshi (1978, 1982) established recurrence relations for exponential distribution with unit mean and were further extended by Balakrishnan and Joshi (1984) for doubly truncated exponential distribution. For linear-exponential distribution, Balakrishnan and Malik(1986) derived the similar type of relations which were extended to doubly truncated linear exponential distribution by Mohie El-Din et al. (1997) and Saran and Pushkarna (1999). Nain (2010 a, b) obtained recurrence relations for ordinary order statistics and k^{th} record values from p^{th} order exponential and generalized weibull distributions, respectively.

The recurrence relations for the moments of generalized order statistics based on non identically distributed random variables were developed by Kamps (1995 a, b). Pawlas and Szynal (2001) obtained recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Saran and Pandey (2004, 2009) obtained recurrence relations for single and product moments of generalized order statistics from linear-exponential and Burr distributions. Saran and Pandey (2011) obtained recurrence relations for marginal and joint moment generating functions of dual (lower) generalized order statistics from inverse Weibull distribution.

In this paper, we have established recurrence relations for single and product moments of GOS from LTLD as well as marginal and joint moment generating functions of GOS from the LTLD. This distribution has many applications in Biology, Epidemiology, Psychology, Computer Technology, Marketing, Energy, Hydrology and Physics. The results so obtained are generalized versions of some of the recurrence relations obtained by Saran and Pandey (2004).

Notations

For n = 1, 2, 3,..., $1 \le r < s \le n,\ k \ge 1$ and $u, v \in \{0,1,2,...\}$, we denote by

i. $\mu^{u}_{r:\ m,\ n,\ k} = E\left(X^{u}(\text{r, n, m, k})\right)$

ii. $\mu^{u,\ v}_{r,\ s:\ m,\ n,\ k} = E\left(X^{u}(\text{r, n, m, k})\ X^{v}(\text{s, n, m, k})\right)$

iii. $\mu^{u}_{r:\ \tilde{m},\ n,\ k} = E\left(X^{u}(\text{r, n, }\tilde{m}\text{, k})\right)$

iv. $\mu^{u,\ v}_{r,\ s:\ \tilde{m},\ n,\ k} = E\left(X^{u}(\text{r, n, }\tilde{m}\text{, k})\ X^{v}(\text{s, n, }\tilde{m}\text{, k})\right)$

4. Recurrence Relations For Single and Product Moments

Case I: $m_1 = m_2 = ... = m_{n-1} = m$

Theorem 1.

For n = 1, 2, 3,..., $1 \le r < s \le n,\ k \ge 1$ and $u, v \in \{0,1,2,...\}$

(a)

$$\mu^{u}_{r:m,n,k} = \gamma_r \sum_{j=0}^{\infty} \frac{\alpha_j}{u+j+1}\left[\mu^{u+j+1}_{r:\ m,\ n,\ k} - \mu^{u+j+1}_{r-1:\ m,\ n,\ k}\right] \quad (4.1)$$

(b)

$$\mu^{u,\ v}_{r,s:m,n,k}(r,s) = \gamma_s \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1}\left[\mu^{u,\ v+j+1}_{r,s:\ m,\ n,\ k} - \mu^{u,\ v+j+1}_{r,s-1:\ m,\ n,\ k}\right] \quad (4.2)$$

Proof (a): The u^{th} order moment of X(r, n, m, k) is given by

$$\mu^{u}_{r:\ m,\ n,\ k} = \frac{c_{r-1}}{(r-1)!} \int_{\beta}^{\infty} x^{u}\left(1-F(x)\right)^{\gamma_r - 1} f(x)\ g_m^{r-1}\left(F(x)\right)dx. \quad (4.3)$$

Substituting f(x) from (2.3) we have

$$\mu^{u}_{r:\ m,\ n,\ k} = \frac{c_{r-1}}{(r-1)!} \sum_{j=0}^{\infty} \alpha_j \int_{\beta}^{\infty} x^{u+j}\left(1-F(x)\right)^{\gamma_r}\ g_m^{r-1}\left(F(x)\right)dx. \quad (4.4)$$

Integrating by parts, taking x^{u+j} as the part to be integrated, we obtain

$$\mu^{u}_{r:\ m,\ n,\ k} = \frac{c_{r-1}}{(r-1)!} \sum_{j=0}^{\infty} \frac{\alpha_j}{u+j+1} \int_{\beta}^{\infty} x^{u+j+1} \Big\{ \gamma_r\left(1-F(x)\right)^{\gamma_r - 1} g_m^{r-1}\left(F(x)\right) f(x) - (r-1)\left(1-F(x)\right)^{\gamma_r + m} g_m^{r-2}\left(F(x)\right) f(x) \Big\} \quad (4.5)$$

On substituting $\gamma_r + m = \gamma_{r-1} - 1$ and $c_{r-1} = \gamma_r c_{r-2}$ in (4.5), we shall derive the recurrence relation as stated in (4.1).

Proof (b). By definition

$$\mu^{u,\ v}_{r,s:\ m,\ n,\ k} = \frac{1}{(r-1)!\,(s-r-1)!} \int_{\beta}^{\infty} x^{u}\left(\left(1-F(x)\right)^{m} f(x)\right) g_m^{r-1}\left(F(x)\right) J\left(x:v,r,s,m\right)dx, \quad (4.6)$$

where,

$$J\left(x:v,r,s,m\right) = c_{s-1} \int_{x}^{\infty} y^{v}\begin{bmatrix} h_m\left(F(y)\right) \\ -h_m\left(F(x)\right)\end{bmatrix}^{s-r-1} \left(1-F(y)\right)^{\gamma_s - 1} f(y)\,dy \quad (4.7)$$

Substituting f(y) from (2.3) we have:

$$J\left(x:v,r,s,m\right) = c_{s-1} \sum_{j=0}^{\infty} \alpha_j \int_{x}^{\infty} y^{v+j}\left[h_m\left(F(y)\right) - h_m\left(F(x)\right)\right]^{s-r-1} \left(1-F(y)\right)^{\gamma_s} f(y)\,dy. \quad (4.8)$$

Integrating by parts, taking y^{v+j} as the part to be integrated, we obtain:

$$J(x:v,r,s,m)=c_{s-1}$$
$$\sum_{j=0}^{\infty}\frac{\alpha_j}{v+j+1}\int_x^{\infty} y^{v+j+1}\left\{\ \gamma_s\begin{bmatrix}h_m(F(y))\\-h_m(F(x))\end{bmatrix}^{s-r-1}\right.$$
$$\times(1-F(y))^{\gamma_s-1}f(y)-(s-r-1)$$
$$\left[h_m(F(y))-h_m(F(x))\right]^{s-r-2}$$
$$\left.\times(1-F(y))^{\gamma_s+m}f(y)\ \right\}dy$$

After using $\gamma_s+m=\gamma_{s-1}-1$ *and* $c_{s-1}=\gamma_s c_{s-2}$, we get:

$$J(x:v,r,s,m)=\gamma_s\sum_{j=0}^{\infty}\frac{\alpha_j}{v+j+1}$$
$$\begin{Bmatrix}J(x:v+j+1,r,s,m)-(s-r-1)\\J(x:v+j+1,r,s-1,m)\end{Bmatrix}.$$

On substituting J(x: v, r, s, m) so obtained in (4.6), we shall derive the recurrence relation as stated in (4.2).

Case II: $\gamma_i\neq\gamma_j$. ∀ i≠j, i, j = 1, 2, …, n - 1.

Lemma 1.

(a)

$$c_{r-1}\sum_{i=1}^{r}a_i(r)\,\gamma_i\,(1-F(x))^{\gamma_i}$$
$$=\gamma_r\left[c_{r-1}\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i}-c_{r-2}\sum_{i=1}^{r-1}a_i(r-1)(1-F(x))^{\gamma_i}\right]\tag{4.9}$$

(b)

$$c_{s-1}\sum_{i=r+1}^{s}a_i^r(s)\,\gamma_i\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}$$
$$=\gamma_s\left[c_{s-1}\sum_{i=r+1}^{s}a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}-c_{s-2}\sum_{i=r+1}^{s-1}a_i^r(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right]\tag{4.10}$$

Proof (a): Differentiating both sides of (3.9), with respect to x, we get:

$$\sum_{i=1}^{r}a_i(r)\,\gamma_i\,(1-F(x))^{\gamma_i-1}=\frac{1}{(r-1)!}$$
$$\left[\gamma_r(1-F(x))^{\gamma_r-1}g_m^{r-1}(F(x))\right.$$
$$\left.-(r-1)(1-F(x))^{\gamma_r}g_m^{r-2}(F(x))g_m^{/}(F(x))\right]$$
$$=\frac{1}{(r-1)!}\left[\gamma_r(1-F(x))^{\gamma_r-1}g_m^{r-1}(F(x))\right.$$
$$\left.-(r-1)(1-F(x))^{\gamma_r+m}g_m^{r-2}(F(x))\right]$$
$$\left(\because\ g_m^{/}(F(x))=(1-F(x))^m\right)$$
$$=\frac{1}{(r-1)!}\begin{bmatrix}\gamma_r(1-F(x))^{\gamma_r-1}g_m^{r-1}(F(x))\\-(r-1)(1-F(x))^{\gamma_r-1^{-1}}g_m^{r-2}(F(x))\end{bmatrix}$$
$$=\frac{1}{1-F(x)}\begin{bmatrix}\gamma_r\sum_{i=1}^{r}a_i(r)(1-F(x))^{\gamma_i}\\-\sum_{i=1}^{r-1}a_i(r-1)(1-F(x))^{\gamma_i}\end{bmatrix},$$

which on using the relation $c_{r-1}=\gamma_r c_{r-2}$ leads to (4.9).

Proof (b): Differentiating both sides of (3.10), with respect to y, we get:

$$\sum_{i=r+1}^{s}a_i^r(s)\,\gamma_i\frac{(1-F(y))^{\gamma_i-1}}{(1-F(x))^{\gamma_i}}=\frac{(1-F(x))^{-(m+1)(s-r-1)}}{(s-r-1)!(1-F(x))^{\gamma_s}}$$
$$\left\{\ \gamma_s(1-F(y))^{\gamma_s-1}\left(h_m(F(y))-h_m(F(x))\right)^{s-r-1}\right.$$
$$-(s-r-1)(1-F(y))^{\gamma_s}$$
$$\left.\left(h_m(F(y))-h_m(F(x))\right)^{s-r-2}h_m^{/}(F(y))\ \right\}$$
$$=\frac{(1-F(x))^{-(m+1)(s-r-1)}}{(s-r-1)!(1-F(x))^{\gamma_s}}$$
$$\left\{\ \gamma_s(1-F(y))^{\gamma_s-1}\left(h_m(F(y))-h_m(F(x))\right)^{s-r-1}\right.$$
$$-(s-r-1)(1-F(y))^{\gamma_s+m}$$
$$\left.\left(h_m(F(y))-h_m(F(x))\right)^{s-r-2}\ \right\}$$
$$\left(\because\ h_m^{/}(F(y))=(1-F(y))^m\right)$$
$$=\left\{\begin{array}{l}\dfrac{\gamma_s(1-F(x))^{-(m+1)(s-r-1)}}{(s-r-1)!(1-F(y))}\\\left(\dfrac{1-F(y)}{1-F(x)}\right)^{\gamma_s}\left(h_m(F(y))-h_m(F(x))\right)^{s-r-1}\\-\dfrac{(1-F(x))^{-(m+1)(s-r-2)}}{(s-r-2)!(1-F(y))}\left(\dfrac{1-F(y)}{1-F(x)}\right)^{\gamma_s-1}\\\left(h_m(F(y))-h_m(F(x))\right)^{s-r-2}\end{array}\right\}$$
$$=\frac{1}{(1-F(y))}\begin{bmatrix}\gamma_s\sum_{i=r+1}^{s}a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\\-\sum_{i=r+1}^{s-1}a_i^r(s-1)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\end{bmatrix},$$

which on using the relation $c_{s-1}=\gamma_s c_{s-2}$ leads to (4.10).

Theorem 2.

For n = 1, 2, 3,..., $1\le r<s\le n$, $k\ge1$ and $u,v\in\{0,1,2,\dots\}$.

(a)

$$\mu_{r:\tilde{m},n,k}^{u}=\gamma_r\sum_{j=0}^{\infty}\frac{\alpha_j}{u+j+1}\left[\mu_{r:\tilde{m},n,k}^{u+j+1}-\mu_{r-1:\tilde{m},n,k}^{u+j+1}\right]\tag{4.11}$$

(b)

$$\mu^{u,v}_{r,s:\tilde{m},n,k} = \gamma_s \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \begin{bmatrix} \mu^{u,\ v+j+1}_{r,s:\tilde{m},n,k} \\ -\mu^{u,\ v+j+1}_{r,s-1:\tilde{m},n,k} \end{bmatrix} \quad (4.12)$$

Proof (a): The uth order moment of $X(r,n,\tilde{m},k)$ is given by:

$$\mu^{u}_{r:\tilde{m},n,k} = c_{r-1} \int_{\beta}^{\infty} x^u \sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i-1} f(x)dx. \quad (4.13)$$

Substituting the value of f(x) from (2.3), we have

$$\mu^{u}_{r:\tilde{m},n,k} = c_{r-1} \sum_{j=0}^{\infty} \alpha_j \left\{ \int_{\beta}^{\infty} x^{u+j} \sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i} dx \right\}. \quad (4.14)$$

Integrating by parts, taking x^{u+j} as the part to be integrated, we obtain:

$$\mu^{u}_{r:\tilde{m},n,k} = c_{r-1} \sum_{j=0}^{\infty} \frac{\alpha_j}{u+j+1} \left\{ \begin{array}{l} \int_{\beta}^{\infty} x^{u+j+1} \\ \sum_{i=1}^{r} a_i(r)\gamma_i (1-F(x))^{\gamma_i-1} f(x) \end{array} \right\} \quad (4.15)$$

After using (4.9), we shall derive the recurrence relation given in (4.11).

Proof (b):

We know that

$$\mu^{u,v}_{r,s:\tilde{m},n,k} = \int_{\beta}^{\infty} x^u \sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i} \frac{f(x)}{1-F(x)} J(x:v,r,s,\tilde{m})dx, \quad (4.16)$$

where

$$J(x:v,r,s,\tilde{m}) = c_{s-1} \int_x^{\infty} y^v \sum_{i=r+1}^{s} a_i^r(s) \left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} \frac{f(y)}{1-F(y)} dy$$

Substituting f(y) from (2.3) we have:

$$J(x:v,r,s,\tilde{m}) = \sum_{j=0}^{\infty} \alpha_j \left\{ c_{s-1} \int_x^{\infty} y^{v+j} \sum_{i=r+1}^{s} a_i^r(s) \left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} dy \right\} \quad (4.17)$$

Integrating by parts, taking y^{v+j} as the part to be integrated, we obtain:

$$J(x:v,r,s,\tilde{m}) = \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \left\{ \begin{array}{l} c_{s-1} \int_x^{\infty} y^{v+j+1} \\ \sum_{i=r+1}^{s} a_i^r(s) \frac{\gamma_i (1-F(y))^{\gamma_i-1}}{(1-F(x))^{\gamma_i}} f(y)dy \end{array} \right\}$$

$$= \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \left\{ \begin{array}{l} y^{v+j+1} \frac{f(y)}{1-F(y)} \\ \int_x^{\infty} \left[c_{s-1} \sum_{i=r+1}^{s} \frac{a_i^r(s) \gamma_i (1-F(y))^{\gamma_i}}{(1-F(x))^{\gamma_i}} \right] dy \end{array} \right\}$$

$$= \gamma_s \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \left\{ \left[c_{s-1} \int_x^{\infty} y^{v+j+1} \sum_{i=r+1}^{s} a_i^r(s) \left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} - c_{s-2} \int_x^{\infty} y^{v+j+1} \sum_{i=r+1}^{s-1} a_i^r(s-1) \left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i} \right] \frac{f(y)}{1-F(y)} dy \right\}$$

[by using (22)]

$$= \gamma_s \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \begin{bmatrix} J(x:v+j+1,r,s,\tilde{m}) \\ -J(x:v+j+1,r,s-1,\tilde{m}) \end{bmatrix} \quad (4.18)$$

On substituting the above expression of $J(x:v,r,s,\tilde{m})$ in (4.16) we get

$$\mu^{u,v}_{r,s:\tilde{m},n,k} = \gamma_s \sum_{j=0}^{\infty} \frac{\alpha_j}{v+j+1} \int_{\beta}^{\infty} x^u \sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i} \frac{f(x)}{1-F(x)} \left\{ \begin{array}{l} J(x:v+j+1,r,s,\tilde{m}) \\ -J(x:v+j+1,r,s-1,\tilde{m}) \end{array} \right\} dx$$

which on using (4.16), leads to (4.12).

5. Conclusion

The relation (2.3) is independent of β .That is, recurrence relations established in theorems 1-2 will work for logistic distribution as well as for half distribution. In the study presented above, we demonstrate the recurrence relations for single and product moments of GOS from LTLD. These results generalize the corresponding results of Kumar (2010) for $p = \infty$.

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