

Paradox Algorithm in Application of a Linear Transportation Problem

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Abstract Paradox seldom occurs in a linear transportation problem, but it is related to the classical transportation problem. For specific reasons of this problem, an increase in the quantity of goods or number of passengers (as used in this paper) to be transported may lead to a decrease in the optimal total transportation cost. Two numerical examples were used for the study. In this paper, an efficient algorithm for solving a linear programming problem was explicitly discussed, and it was concluded that paradox does not exist in the first set of data, while paradox exists in the second set of data. The Vogel's Approximation Method (VAM) was used to obtain the initial basic feasible solution via the Statistical Software Package known as TORA. The first set of data revealed that paradox does not exist, while the second set of data showed that paradox exists. The method however gives a step by step development of the solution procedure for finding all the paradoxical pair in the second set of data.

Keywords: transportation paradox, paradoxical range of flow, transportation problem, linear programming, paradoxical pair, VAM

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1. Introduction

Paradox occurs when a transportation problem admits of a total cost which is lower than the optimum and is attainable by shipping larger quantities of goods over the same routes that were previously designated as optimal. This phenomenon which does not occur regularly was discovered by Szwarz (1971). It is obvious that many researchers have discovered independently from each other the following behavior of the transportation problem. In certain cases of the Transportation Problem (TP), an increase in the supplies and demands may lead to a decrease in the optimal transportation cost. In other words, by moving bigger amount of goods around, one may save a lot of money. This surely sounds paradoxical (Deineko, et al.; 2003).

The classical transportation problem is the name of a mathematical model which has a special mathematical structure. The mathematical formulation of a large number of problems conforms to this special structure. Hitchcock (1941) originally developed the basic linear transportation problem. Charnes et al (1953) developed the stepping stone methods which provide an alternative way of determining the simplex method information. Dantzig (1963) used the simplex method to the transportation problem as the primal simplex transportation method.

Appa (1973) also developed the solution procedure for solving the transportation problem and its variants. Klingman and Russel (1974 and 1975) introduced a specialized method for solving a transportation problem with several additional linear constraints. Hadley (1987) gave the detailed solution procedure for solving linear transportation problem. Till date, several researchers studied extensively to solve cost minimizing transportation problem in various ways.

In some situations, if we obtain more flow with lesser cost than the flow corresponding to the optimum cost then we say paradox occurs. Charnes and Klingman (1971), Szwarz (1973), Adlakhia and Kowalski (1998) and Storoy (2007) considered the paradoxical transportation problem. Gupta et al (1993) considered a paradox in linear fractional transportation problem with mixed constraints. Joshi and Gupta (2010) studied paradox in linear plus fractional transportation problem. In the early day of linear programming problem some of the pioneers observed paradox but by whom no one knows.

In this paper we present a method for solving transportation problem with linear constraints. Thereby, we state the sufficient condition of existence of paradox, paradoxical range of flow and paradoxical flow for a specified flow in such type of linear transportation problem. We also justify the theory by illustrating a numerical example.

2. Definitions

1. Paradox in a transportation problem: in a transportation problem if we can obtain more flow (F^1) with lesser cost (Z^1) than the optimum flow (F^0) corresponding to the optimum cost (Z^0) i.e. $F^1 > F^0$ and $Z^1 < Z^0$, then we say that a paradox occurs in a transportation problem.
2. Cost-flow pair: if the value of the objective function is Z^0 and the flow is F^0 corresponding to the feasible solution X^0 of a transportation problem, then the pair (Z^0, F^0) is called the cost-flow pair corresponding to the feasible solution X^0 .
3. Paradoxical pair: A cost-flow pair, (Z, F) of an objective function is called paradoxical pair if $Z < Z^0$ and $F > F^0$ where Z^0 is the optimum cost and F^0 is the optimum flow of the transportation problem.
4. Best paradoxical pair: The paradoxical pair (Z^*, F^*) is called the best paradoxical pair of a transportation problem if for all paradoxical pair (Z, F) , either $Z^* < Z$ or $Z^* = Z$ but $F^* > F$.
5. Paradoxical range of flow: if F^0 be the optimum flow and F^* be the flow corresponding to the best paradoxical pair of a transportation problem then $[F^0, F^*]$ is called paradoxical range of flow.

3. Review of Related Literature

The transportation paradox is, however, hardly mentioned at all in any of the great number of textbooks and teaching materials where the transportation problem is treated. Apparently, several researchers have discovered the paradox independently from each other. But most papers on the subject refer to the paper by Charines and Klingman (1971) and Szwarc (1971) as the initial papers. In Charines and Klingman (1971) name it the more-for-less paradox and they write: "The paradox was first observed in the early days of linear programming history (by whom no one knows) and has been part of the folklore known to some (e.g. A. Charnes and Cooper; 1953) but unknown to the great majority of workers in the field of linear programming.

Arora and Ahuja (2010) carried out a research work in paradox on a fixed charge transportation problem. In their findings, a paradox arises when the fixed charge transportation problem admits of a total cost which is lower than the optimum cost, by transporting larger quantities of goods over the same routes. A sufficient condition for the existence of a paradox is established. Paradoxical Range of flow is obtained for any given flow in which the corresponding objective function value is less than the optimum value of the fixed charge transportation problem.

Ekezie et al (2013) carried out a research on the determination of paradoxical pairs in a linear transportation problem. In their study, an efficient algorithm for solving a linear programming problem was discussed, and it was concluded that paradox exists. The North-West Corner method was used to obtain the initial basic feasible solution for the optimal solution. They also used the algorithm discussed to develop a step by step solution procedure for finding all the paradoxical pairs.

According to Appa (1973), the transportation paradox is known as Doigs paradox at the London School of Economics, named after Alison Doig who used it in exams etc around 1959 (Doig did not publish any paper on it).

Since the transportation paradox seems not to be known to the majority of those who are working with (or teaching) the transportation problem, one may be tempted to believe that this phenomenon is only an academic curiosity which will most probably not occur in any practical situation. But that seems not to be true. Experiments done by Finke (1978), with randomly generated instances of the transportation problem of size 100×100 and allowing additional shipments (post-optimal) show that the paradoxical properties. More precisely, the average cost reductions achieved are reported to be 18.6% with total additional shipments of 20.5%.

Ekezie et al (2013) carried out a research on the Paradox sum of a linear and a linear fractional transportation problem using data that were collected from a secondary source. In the study, a transportation problem with an objective function as the sum of a linear and linear fractional function was considered. A paradoxical situation arises in the sum of a linear and linear fractional transportation problem, when value of the objective function falls below the optimal value and this lower value is attainable by transporting larger number of passengers. An algorithm is proposed for finding initial basic feasible solution for the sum of a linear and linear fraction transportation problem and a sufficient condition for the existence of a paradoxical solution is established.

In a recent paper (2003), Deineko et al developed necessary and sufficient conditions for a cost matrix C to be immune against the transportation paradox. As well see in the next section, these conditions are rather restrictive, supporting the observations by Finke.

4. Problem Formulation

In this paper, we consider the following transportation problem:

Let the transportation problem consists of m sources and n destinations, where

x_{ij} = the amount of product transported from the i th source to the j th destination,

c_{ij} = the cost involved in transporting per unit product from the i th source to the j th destination,

a_i = the number of units available at the i th source,

b_j = the number of units required at the j th destination.

In this paper, we consider the cost minimizing linear transportation problem as:

$$P_1 : \text{Min}Z = \sum_{i=1}^m \sum_{j=1}^n c_{ij}x_{ij}$$

subject to the constraints

$$\sum_{j=1}^n x_{ij} = a_i; \forall i \in I = (1, 2, \dots, m)$$

$$\sum_{i=1}^m x_{ij} = b_j; \forall j \in J = (1, 2, \dots, m)$$

$$\text{and } x_{ij} \geq 0 \forall (i, j) \in I \times J$$

Let $x^0 = \{x_{ij}^0 \mid (i, j) \in I \times J\}$ be a basic feasible solution corresponding to the basis B of the problem P_1 and the value of the objective function Z^0 corresponding to the basic feasible solution X^0 is

$$Z^0 = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}^0 \text{ (say)}$$

Let F^0 be the corresponding flow.

$$\text{Then } F^0 = \sum_{i \in I} a_i = \sum_{j \in J} b_j.$$

Now we consider the dual variables u_i for $i \in I$ and $v_j \in J$ such that $u_i + v_j = c_{ij}$ corresponding to the basis B.

Also $\forall (i, j) \notin B$, let

$$c'_{ij} = (u_i + v_j) - c_{ij}$$

If $c'_{ij} < 0 \forall (i, j) \notin B$ then the solution is optimum.

Theorem: the sufficient condition for the existence of paradoxical solution of (P_1) is that if \exists at least one cell $(r, s) \notin B$ in the optimum table of (P_1) where a_r and b_s are replaced by $a_r + l$ and $b_s + l$ respectively ($l > 0$) then $(u_r + v_s) < 0$.

Proof: Let Z^0 be the value of the objective function and F^0 be the optimum flow corresponding to the optimum solution (X^0) of problem (P_1) . The dual variables u_i and v_j are given by

$$u_i + v_j = c_{ij}, \forall (i, j) \in B$$

Then

$$\begin{aligned} Z^0 &= \sum_i \sum_j c_{ij} x_{ij}^0 = \sum_i \sum_j (u_i + v_j) x_{ij}^0 \\ &= \sum_i \left(\sum_j x_{ij}^0 \right) u_i + \sum_j \left(\sum_i x_{ij}^0 \right) v_j \\ &= \sum_i a_i u_i + \sum_j b_j v_j \end{aligned}$$

and

$$F^0 = \sum_i a_i = \sum_j b_j$$

Now, let \exists at least one cell $(r, s) \notin B$, where a_r are replaced by $a_r + l$ and $b_s + l$, respectively ($l > 0$) in such a way that the optimum basis remains same, then the value of the objective function \hat{Z} is given by

$$\begin{aligned} \hat{Z} &= \left[\sum_{i \neq r} a_i u_i + \sum_{j \neq s} b_j v_j + u_r (a_r + l) + v_s (b_s + l) \right] \\ &= \left[Z^0 + l(u_r + v_s) \right] \end{aligned}$$

The new flow \hat{F} is given by

$$\hat{F} = \sum_i a_i + l = \sum_j b_j + l = F^0 + l$$

$$\hat{F} - F^0 = l > 0$$

Therefore for the existence of paradox we must have $\hat{Z} - Z^0 < 0$. Hence the sufficient condition for the existence of paradox is that \exists at least one cell $(r, s) \notin B$ in the optimum table of such that if a_r and b_s are replaced by $a_r + l$ and $b_s + l$ ($l > 0$) then $l(u_r + v_s) < 0$, i.e. $(u_r + v_s) < 0$.

Now we state the following algorithm to find all the paradoxical pair of the problem (P_1) .

Algorithm:

Step 1: Find the cost-flow pair (Z^0, F^0) for the optimum solution X^0 .

Step 2: $i = 1, \bar{F}$

Step 3: Find all cells $(r, s) \notin B$ such that $(u_r + v_s) < 0$ if it exists otherwise go to step 8.

Step 4: Among all cells $(r, s) \notin B$ satisfying step 3 find min flow for $l = 1$ which enter into the existing basis whose corresponding cost is minimum. Let (Z^1, F^1) be the new cost flow pair corresponding to the optimum solution X^1 .

Step 5: Write (Z^i, F^i) .

Step 6: $i = i + 1$.

Step 7: go to step 3

Step 8: We write the best paradoxical pair $(Z^*, F^*) = (Z^i, F^i)$ for the optimum solution $X^* = X^i$.

Step 9: End.

This algorithm gives all the paradoxical pairs. From these pairs we can find the paradoxical pair for a specified flow () also.

5. Data Analysis

In this paper, we shall use three sets of data to enable us achieve our primary aim. Firstly, we shall examine the data gotten from ABC Transport Company, owerri Imo State Nigeria. ABC Transport Company Ltd has many types of buses that ply different routes in Nigeria. In this study, we shall restrict the work to only 6 different types of buses that ply 4 different routes in Nigeria. [Table 1](#) shows the expected number of passengers each type of bus can carry in a day, and the cost of transporting a passenger from Owerri to the different routes in Nigeria by each of the buses.

Table 1.

		Routes				
		Lagos	Abuja	Jos	Kaduna	Supply
Bus types	Coach Bus 1	5820	5960	5850	5640	165
	Coach Bus 2	5680	5720	5720	5580	102
	Coach Bus 3	5590	5650	5630	5410	66
	Coach Bus 4	5480	5680	5890	5720	42
	Smart Coach Bus	6850	6520	6870	6630	33
	Sprinter Bus	6980	6860	6900	6820	15
	Demand	168	134	76	45	423

Solving the transportation problem in [Table 1](#), the result is displayed in [Table 2](#).

Table 2.

	D ₁	D ₂	D ₃	D ₄	S _i	U _i
S ₁	5820 44	5960	5850 76	5640 45	165	0
S ₂	5680 16	5720	5720	5580	102	-140
S ₃	5590 66	5650	5630	5410	66	-230
S ₄	5480 42	5680	5890	5720	42	-340
S ₅	6850	6520 33	6870	6630	33	660
S ₆	6980	6860 15	6900	6820	15	1000
D _j	168	134	76	45	423	
V _j	5820	5860	5850	5640		

The total cost of transportation is ₦2, 454,440. We then check the sign of (U_r + V_s), where (r, s) ∉ B in Table 2, we observe that all the U_rs + V_s > 0. With this result, we can conclude that paradox does not exist, which implies that the algorithm discussed in this paper will be irrelevant to this set of data.

Let us now consider the second set of data used for this research, which was extracted from Opara (2009), Introduction to Operation Research, Exercises 3 page 28. The estimated supply capacities of the six warehouses, the demand requirements at the seven markets and the transportation cost of each product are given in Table 3.

Table 3.

	D ₁	D ₂	D ₃	D ₄	D ₅	D ₆	D ₇	S _i
S ₁	9	7	3	6	8	14	10	60
S ₂	8	6	4	12	4	5	5	39
S ₃	5	7	1	3	7	12	4	33
S ₄	5	5	5	7	6	7	9	20
S ₅	7	1	5	11	7	8	4	13
S ₆	5	6	6	3	6	7	3	8
d _j	49	42	34	20	17	6	5	173

Solving the above problem using the Vogel's Approximation Method (VAM), the optimal transportation table is presented in Table 4.

Table 4.

	D ₁	D ₂	D ₃	D ₄	D ₅	D ₆	D ₇	S _i	U _i
S ₁	9	7	3	6	8	14	10	60	0
S ₂	8	6	4	12	4	5	5	39	-1
S ₃	5	7	1	3	7	12	4	33	-3
S ₄	5	5	5	7	6	7	9	20	-3
S ₅	7	1	5	11	7	8	4	13	-6
S ₆	5	6	6	3	6	7	3	8	-3
d _j	49	42	34	20	17	6	5	173	
v _j	8	7	3	6	5	6	6		

The total cost is 759. We then check the sign of (U_r + V_s), where (r, s) ∉ B in Table 4, we observe that U₅ + V₃ = -3 < 0, and U₅ + V₅ = -1 < 0. This implies that Paradoxical pair of the problem (P₁) exists. So applying the algorithm discussed in this paper, we have;

Applying Step 1: The cost-flow pair is (Z⁰, F⁰) = (759, 173) corresponding to the optimum solution X = {x₁₂ = 13, x₁₃ = 34, x₁₄ = 13, x₂₂ = 16, x₂₅ = 17, x₂₆ = 6, x₃₁ = 26, x₃₄ = 7, x₄₁ = 20, x₅₂ = 3, x₆₁ = 3, x₆₇ = 5}.

Applying step 2: set i=1

Applying step 3: Now we check the sign of (U_r + V_s) and we obtain for the non-basic cells (5, 3) and (5, 5), the sign that is negative.

Applying step 4: For l = 1

For the cell (5, 3)

Table 5.

	D ₁	D ₂	D ₃	D ₄	D ₅	D ₆	D ₇	S _i	U _i
S ₁	9	7	3	6	8	14	10	60	0
S ₂	8	6	4	12	4	5	5	39	-1
S ₃	5	7	1	3	7	12	4	33	-3
S ₄	5	5	5	7	6	7	9	20	-3
S ₅	7	1	5	11	7	8	4	14	-6
S ₆	5	6	6	3	6	7	3	8	-3
d _j	49	42	35	20	17	6	5	174	
v _j	8	7	3	6	5	6	6		

The total cost of this transportation problem is 756.

For cell (5, 5), the transportation table is presented in Table 6.

Table 6.

	D ₁	D ₂	D ₃	D ₄	D ₅	D ₆	D ₇	S _i	U _i
S ₁	9	7	3	6	8	14	10	60	0
S ₂	8	6	4	12	4	5	5	39	-1
S ₃	5	7	1	3	7	12	4	33	-3
S ₄	5	5	5	7	6	7	9	20	-3
S ₅	7	1	5	11	7	8	4	14	-6
S ₆	5	6	6	3	6	7	3	8	-3
d _j	49	42	34	20	18	6	5	174	
v _j	8	7	3	6	5	6	6		

The total cost of this transportation problem is 758.

The min cost = min{756, 758} = 756.

Hence l = 1 enters in the optimum basis from the cell (5, 3) and corresponding table is Table 5, the corresponding paradoxical pair (Z', F') = (756, 174).

Employing steps 6 and 7. Then repeating this process, the next table is

Table 7.

	D ₁	D ₂	D ₃	D ₄	D ₅	D ₆	D ₇	S _i	U _i
S ₁	9	7	3	6	8	14	10	60	0
S ₂	8	6	4	12	4	5	5	39	-1
S ₃	5	7	1	3	7	12	4	33	-3
S ₄	5	5	5	7	6	7	9	20	-3
S ₅	7	1	5	11	7	8	4	15	-6
S ₆	5	6	6	3	6	7	3	8	-3
d _j	49	42	36	20	17	6	5	175	
v _j	8	7	3	6	5	6	6		

Now repeating this process, the final table is presented in [Table 8](#).

Table 8.

	D ₁	D ₂	D ₃	D ₄	D ₅	D ₆	D ₇	S _i	U _i
S ₁	9	7 0	3 47	6 13	8	14	10	60	0
S ₂	8	6 16	4	12	4 17	5 6	5	39	-1
S ₃	5 26	7	1	3 7	7	12	4	33	-3
S ₄	5 20	5	5	7	6	7	9	20	-3
S ₅	7	1 26	5	11	7	8	4	26	-6
S ₆	5 3	6	6	3	6	7	3 5	8	-3
d _j	49	53	36	20	17	6	5	186	
v _j	8	7	3	6	5	6	6		

Hence, from the above table, the corresponding paradoxical pair $(Z', F') = (720, 186)$.

Applying step 8: The best paradoxical pair is $(Z^*, F^*) = (720, 186)$ corresponding to the optimum solution is $X^* = \{x_{12} = 0; x_{13} = 47, x_{14} = 13, x_{22} = 16, x_{25} = 17, x_{26} = 6, x_{31} = 26, x_{34} = 7, x_{41} = 20, x_{52} = 26, x_{61} = 3, x_{67} = 5\}$ and the paradoxical range of flow is $[F^0, F^*] = [173, 186]$. Thus, all the paradoxical pairs are $\{(756, 174), (753, 175), (750, 176), (747, 177), (744, 178), (741, 179), (738, 180), (735, 181), (732, 182), (729, 183), (726, 184), (723, 185)$ and $(720, 186)$.

Let us consider the third set of data from this hypothetical example for more clarification of the algorithm. The estimated supply capacities of the five warehouses, the demand requirements at the five markets and the transportation cost of each product are given in [Table 8](#).

Table 9.

(i, j)	M ₁	M ₂	M ₃	M ₄	M ₅	S _i	U _i
W ₁	6	4	8	4	2	45	0
W ₂	2	3	2	5	3	35	-4
W ₃	3	4	1	7	1	30	-3
W ₄	5	3	4	1	2	22	-3
W ₅	7	2	5	4	7	10	-2
D _j	44	36	27	25	10	142	

Solving the above problem using VAM via TORA Statistical Software Package, the optimal transportation table is presented in [Table 10](#).

Table 10.

(i, j)	M ₁	M ₂	M ₃	M ₄	M ₅	S _i	U _i
W ₁	6	4	8	4	2	45	0
W ₂	2	3	2	5	3	35	-4
W ₃	3	4	1	7	1	30	-3
W ₄	5	3	4	1	2	22	-3
W ₅	7	2	5	4	7	10	-2
D _j	44	36	27	25	10	142	
V _j	6	4	4	4	2		

The total cost is 320. Again, we then check the sign of $(U_r + V_s)$, where $(r, s) \notin B$ in [Table 10](#), we observe that $U_2 + V_5 = -2 < 0$, $U_3 + V_5 = -1 < 0$, and $U_4 + V_5 = -1 < 0$. So, paradoxical pairs exist.

Applying Step 1: The cost-flow pair is $(Z^0, F^0) = (320, 142)$ corresponding to the optimum solution $X = \{x_{11} = 6, x_{12} = 26, x_{14} = 3, x_{15} = 10, x_{21} = 35, x_{31} = 3, x_{33} = 27, x_{44} = 22, x_{52} = 10\}$.

Applying step 2: set $i=1$

Applying step 3: Now we check the sign of $(U_r + V_s)$ and we obtain for the non-basic cells $(2,5)$, $(3,5)$ and $(4,5)$, the sign that is negative.

Applying step 4: For $l = 1$

For the cell $(2, 5)$

Table 11.

(i, j)	M ₁	M ₂	M ₃	M ₄	M ₅	S _i	U _i
W ₁	6	4	8	4	2	45	0
W ₂	2	3	2	5	3	36	-4
W ₃	3	4	1	7	1	30	-3
W ₄	5	3	4	1	2	22	-3
W ₅	7	2	5	4	7	10	-2
D _j	44	36	27	25	11	143	
V _j	6	4	4	4	2		

The total cost of this transportation problem is 318.

For cell $(3, 5)$, the transportation tableau is presented in [Table 12](#).

Table 12.

(i, j)	M ₁	M ₂	M ₃	M ₄	M ₅	S _i	U _i
W ₁	6	4	8	4	2	45	0
W ₂	2	3	2	5	3	35	-4
W ₃	3	4	1	7	1	31	-3
W ₄	5	3	4	1	2	22	-3
W ₅	7	2	5	4	7	10	-2
D _j	44	36	27	25	11	143	
V _j	6	4	4	4	2		

The total cost of this transportation problem is 319.

For the cell $(4, 5)$, the transportation tableau is presented in [Table 13](#).

Table 13.

(i, j)	M ₁	M ₂	M ₃	M ₄	M ₅	S _i	U _i
W ₁	6	4	8	4	2	45	0
W ₂	2	3	2	5	3	35	-4
W ₃	3	4	1	7	1	31	-3
W ₄	5	3	4	1	2	22	-3
W ₅	7	2	5	4	7	10	-2
D _j	44	36	27	25	11	143	
V _j	6	4	4	4	2		

The total cost is 319.

The min cost = $\min \{318, 319, 319\} = 318$.

Hence $l = 1$ enters in the optimum basis from the cell (2, 5) and corresponding table is [Table 10](#), the corresponding paradoxical pair $(Z', F') = (318, 143)$.

Employing steps 6 and 7. Then repeating this process again, the next tableau is

Table 14.

(i, j)	M ₁	M ₂	M ₃	M ₄	M ₅	S _i	U _i
W ₁	6 4	4 26	8	4 3	2 12	45	0
W ₂	2 37	3	2	5	3	35	-4
W ₃	3 3	4	1 27	7	1	31	-3
W ₄	5	3	4	1 22	2	22	-3
W ₅	7	2 10	5	4	7	10	-2
D _j	44	36	27	25	12	144	
V _j	6	4	4	4	2		

Now repeating this process, the final table is presented in [Table 15](#).

Table 15.

(i, j)	M ₁	M ₂	M ₃	M ₄	M ₅	S _i	U _i
W ₁	6 0	4 26	8	4 3	2 16	45	0
W ₂	2 41	3	2	5	3	35	-4
W ₃	3 3	4	1 27	7	1	31	-3
W ₄	5	3	4	1 22	2	22	-3
W ₅	7	2 10	5	4	7	10	-2
D _j	44	36	27	25	16	148	
V _j	6	4	4	4	2		

Hence, from the above table, the corresponding paradoxical pair $(Z', F') = (308, 148)$.

Applying step 8: The best paradoxical pair is $(Z^*, F^*) = (308, 148)$ corresponding to the optimum solution is $X^* = \{x_{11} = 0; x_{12} = 26, x_{14} = 3, x_{15} = 16, x_{21} = 41, x_{31} = 3, x_{33} = 27, x_{44} = 22, x_{52} = 10\}$ and the paradoxical range of flow is $[F^0, F^*] = [142, 148]$. Thus, all the paradoxical pair are $\{(318, 143), (316, 144), (314, 145), (312, 146), (310, 147), \text{ and } (308, 148)\}$.

6. Conclusion

In this paper, an attempt has been made to discuss an efficient statistical algorithm for computing paradox in a linear transportation problem if paradox does exist. The algorithm gives step by step development of the solution procedure for finding all the paradoxical pair where

paradox exists, well understanding. The statistical software package known as “TORA” was used to obtain the optimal solution before adopting the algorithm of paradoxical pairs.

Future research should be done on a related work like in this paper, and the author should try to develop a program that will enable one to obtain the best paradoxical pairs instead of solving it manually, and as well the paradoxical range of flow.

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