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Generalised Common Fixed Point Theorems of A-Compatible and S-Compatible Mappings

Shahidur Rahman, Yumnam Rohen*, M. Popeshwar Singh

National Institute of Technology Manipur, Imphal, India
*Corresponding author: ymnehor2008@yahoo.com

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Abstract In this paper we prove a common fixed point theorem of four self mappings satisfying a generalized inequality using the concept of A-compatible and S-compatible mappings. Our result generalizes many earlier related results in the literature.

Keywords: *common fixed point, complete metric space, compatible mappings, compatible mappings of type (A), A-compatible mappings, S-compatible mappings*

1. Introduction

The first important result in the theory of fixed point of compatible mappings was obtained by Gerald Jungck in 1986 [2] as a generalization of commuting mappings. In 1993 Jungck, Murthy and Cho [3] introduced the concept of compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. Pathak and Khan [6] introduced the concept of A-compatible and S-compatible by splitting the definition of compatible mappings of type (A). Fixed point results of compatible mappings are found in [1-8].

Sharma and Sahu [8] proved the following theorem.

THEOREM 1.1 Let A, S and T be three continuous mappings of a complete metric space (X, d) into itself satisfying the following conditions:

(i) A commutes with S and T respectively

(ii) $S(X) \subseteq A(X)$ and $T(X) \subseteq A(X)$

(iii) $[d(Sx, Tx)]^2 \le a_1 d(Ax, Sx)d(Ay, Ty)+a_2 d(Ay, Sx)d(Ax, Ty)+a_3 d(Ax, Sx)d(Ax, Ty) +a_4 d(Ay, Ty)d(Ay, Sx)+a_5 d^2(Ax, Ay)$

For all $x, y \in X$, where $a_i \ge 0$, $i = 1, 2, 3, 4, 5$ and $a_1+a_4+a_5 < 1$, $2a_1+3a_3+2a_5<2$.

Then A, S and T have a unique common fixed point in X.

Murthy [6] pointed out that the constraints taken by Sharma and Sahu in condition (iii) of theorem 1.1 is not true and suggested the corrected replacement as max $\{a_1+2a_3+a_5, a_1+2a_4+a_5, a_2+a_5\} < 1$ and proved a new fixed point theorem.

The aim of this paper is to prove a common fixed point theorem of S-compatible mappings in metric space by considering four self mappings. Further we give another common fixed point theorem of A-compatible mappings.

2. Preliminaries

Following are definitions of types of compatible mappings.

Definition 2.1 [2]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 2.2 [3]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be compatible of type (A) if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 2.3 [5]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be A-compatible if $\lim_{n\to\infty} d(ASx_n, SSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 2.4 [5]: Let A and S be mappings from a complete metric space X into itself. The mappings A and S are said to be S-compatible if $\lim_{n\to\infty} d(SAx_n, AAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that for $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Proposition 2.5 [6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is A-compatible on X and $St = At$ for $t \in X$, then $ASt = SSt$.

Proposition 2.6 [6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is S-compatible on X and $St = At$ for $t \in X$, then $SAt = AAt$.

Proposition 2.7 [6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is

A-compatible on X and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for $t \in X$, then $SSx_n \to At$ if A is continuous at t.

Proposition 2.8 [6]: Let A and S be mappings from a complete metric space (X, d) into itself. If a pair (A, S) is S-compatible on X and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for $t \in X$, then $AAx_n \to St$ if S is continuous at t.

Now we prove the following theorem.

LEMMA 2.9 Let A, B, S and T be mapping from a metric space (X, d) into itself satisfying the following conditions:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2) $[d(Ax, Bx)]^2 \le a_1d(Ax, Sx)d(By, Ty)+a_2d(By, Sx)d(Ax, Ty)+a_3d(Ax, Sx)d(Ax, Ty)+a_4d(By, Ty)d(By, Sx) +a_5d^2(Sx, Ty)$

where $a_1+ a_2 +2a_3 +a_4+ a_5< 1$ and $a_1, a_2, a_3, a_4, a_5 \ge 0$

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$y_{2n+1}=Tx_{2n+1}=Ax_{2n}$ and $y_{2n}= Sx_{2n}= Bx_{2n-1}$

then the sequence $\{y_n\}$ is Cauchy sequence in X.

Proof. By condition (2) and (3), we have

$$\begin{aligned}\left[d\left(y_{2n+1}, y_{2n}\right)\right]^2 &= \left[d\left(Ax_{2n}, Bx_{2n-1}\right)\right]^2 \\ &\le a_1 d\left(Ax_{2n}, Sx_{2n}\right) d\left(Bx_{2n-1}, Tx_{2n-1}\right) \\ &\quad + a_2 d\left(Bx_{2n-1}, Sx_{2n}\right) d\left(Ax_{2n}, Tx_{2n-1}\right) \\ &\quad + a_3 d\left(Ax_{2n}, Sx_{2n}\right) d\left(Ax_{2n}, Tx_{2n-1}\right) \\ &\quad + a_4 d\left(Bx_{2n-1}, Tx_{2n-1}\right) d\left(Bx_{2n-1}, Sx_{2n}\right) \\ &\quad + a_5 d^2\left(Sx_{2n}, Tx_{2n-1}\right) \\ &= a_1 d\left(y_{2n+1}, y_{2n}\right) d\left(y_{2n}, y_{2n-1}\right) \\ &\quad + a_2 d\left(y_{2n}, y_{2n}\right) d\left(y_{2n+1}, y_{2n-1}\right) \\ &\quad + a_3 d\left(y_{2n+1}, y_{2n}\right) d\left(y_{2n+1}, y_{2n-1}\right) \\ &\quad + a_4 d\left(y_{2n}, y_{2n-1}\right) d\left(y_{2n}, y_{2n}\right) \\ &\quad + a_5 d^2\left(y_{2n}, y_{2n-1}\right)\end{aligned}$$

$$\begin{aligned}\left[d\left(y_{2n+1}, y_{2n}\right)\right]^2 &\le a_1 d\left(y_{2n+1}, y_{2n}\right) d\left(y_{2n}, y_{2n-1}\right) \\ &\quad + a_3 d^2\left(y_{2n+1}, y_{2n}\right) \\ &\quad + a_3 d\left(y_{2n+1}, y_{2n}\right) d\left(y_{2n}, y_{2n-1}\right) \\ &\quad + a_5 d^2\left(y_{2n}, y_{2n-1}\right)\end{aligned}$$

$$(1-a_3)\left\{\frac{d(y_{2n+1}, y_{2n})}{d(y_{2n}, y_{2n-1})}\right\}^2 \le (a_1 + a_3)\left\{\frac{d(y_{2n+1}, y_{2n})}{d(y_{2n}, y_{2n-1})}\right\} + a_5$$

$$\Rightarrow \lambda^2 - \lambda B - C \le 0$$

Where $\lambda = \dfrac{d(y_{2n+1}, y_{2n})}{d(y_{2n}, y_{2n-1})}$

$$B = \frac{a_1 + a_3}{1 - a_3}$$

$$C = \frac{a_5}{1 - a_3}$$

Since $a_1+ a_2 +2a_3 +a_4+ a_5< 1$ and $a_1, a_2, a_3, a_4, a_5 \ge 0$.

In order to satisfy the inequation, one value of λ will be positive and the other will be negative. We also note that the sum and product of the two values of λ is less than 1 and -1 respectively. Neglecting the negative value, we have $\dfrac{d(y_{2n+1}, y_{2n})}{d(y_{2n}, y_{2n-1})} < p$ where $0<p<1$.

$$d(y_{2n+1}, y_{2n}) < pd(y_{2n}, y_{2n-1})$$

Hence $\{y_n\}$ is Cauchy sequence.

3. Main Results

We prove the following theorem.

THEOREM 3.1: Let A, B, S and T be self maps of a complete metric space (X, d) satisfying the following conditions:

(1) A (X) ⊆ T(X) and B(X) ⊆ S(X)

(2) [d(Ax, Bx)]2 ≤ a1d(Ax, Sx)d(By, Ty)+a2d(By, Sx)d(Ax, Ty)+a3d(Ax, Sx)d(Ax, Ty)+a4d(By, Ty)d(By, Sx) +a5d2(Sx, Ty)

where a1+ a2 +2a3 +a4+ a5< 1 and a1, a2, a3, a4, a5 ≥ 0

(3) Let x0 ∈ X then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1}$$

then the sequence $\{y_n\}$ is Cauchy sequence in X.

(4) One of A, B, S or T is continuous.

(5) $[A, S]$ and $[B, T]$ are S-compatible mappings on X.

Then A, B, S and T have a unique common fixed point in X.

Proof: By lemma 2.9, $\{y_n\}$ is Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $\lim y_n = z$ as $n \to \infty$. Consequently subsequences Ax_{2n}, Sx_{2n}, Bx_{2n-1} and Tx_{2n+1} converges to z.

Let S be a continuous mapping. Since A and S are S-compatible mappings on X, then by proposition 2.8., we have $AAx_{2n} \to Sz$ and $SAx_{2n} \to Sz$ as $n \to \infty$.

Now by condition (2) of lemma 2.9, we have

$$\begin{aligned}d^2\left(AAx_{2n}, Bx_{2n-1}\right) &\le a_1 d\left(AAx_{2n}, SAx_{2n}\right) d\left(Bx_{2n-1}, Tx_{2n-1}\right) \\ &\quad + a_2 d\left(Bx_{2n-1}, SAx_{2n}\right) d\left(AAx_{2n-1}, Tx_{2n-1}\right) \\ &\quad + a_3 d\left(AAx_{2n}, SAx_{2n}\right) d\left(AAx_{2n}, Tx_{2n-1}\right) \\ &\quad + a_4 d\left(Bx_{2n-1}, Tx_{2n-1}\right) d\left(Bx_{2n-1}, SAx_{2n}\right) \\ &\quad + a_5 d^2\left(SAx_{2n}, Tx_{2n-1}\right)\end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned}d^2\left(Sz, z\right) &\le a_1 d\left(Sz, Sz\right) d\left(z, z\right) + a_2 d\left(z, Sz\right) d\left(Sz, z\right) \\ &\quad + a_3 d\left(Sz, Sz\right) d\left(Sz, z\right) + a_4 d\left(z, z\right) d\left(z, Sz\right) \\ &\quad + a_5 d^2\left(Sz, z\right)\left[d\left(Sz, z\right)\right]^2 \\ &\le \left(a_2 + a_5\right)\left[d\left(Sz, z\right)\right]^2,\end{aligned}$$

which is a contradiction. Hence $Sz = z$,

Now

$$\left[d\left(Az,Bx_{2n-1}\right)\right]^2 \le a_1 d\left(Az,Sz\right)d\left(Bx_{2n-1},Tx_{2n-1}\right) + a_2 d\left(Bx_{2n-1},Sz\right)d\left(Az,Tx_{2n-1}\right) + a_3 d\left(Az,Sz\right)d\left(Az,Tx_{2n-1}\right) + a_4 d\left(Bx_{2n-1},Tx_{2n-1}\right)d\left(Bx_{2n-1},Sz\right) + a_5 d^2\left(Sz,Tx_{2n-1}\right)$$

Letting $n\to\infty$, we have $[d(Az, z)]^2 \le a_3[d(Az, z)]^2$. Hence $Az = z$.

Now since $Az = z$, by condition (1), $z \in T(X)$. Also T is self map of X so there exists a point $u \in X$ such that $z = Az = Tu$. More over by condition (2), we obtain,

$$\left[d\left(z,Bu\right)\right]^2 = \left[d\left(Az,Bu\right)\right]^2 \le a_1 d\left(Az,Sz\right)d\left(Bu,Tu\right) + a_2 d\left(Bu,Sz\right)d\left(Az,Tu\right) + a_3 d\left(Az,Sz\right)d\left(Az,Tu\right) + a_4 d\left(Bu,Tu\right)d\left(Bu,Sz\right) + a_5 d^2\left(Sz,Tu\right)$$

i.e., $[d(z, Bu)]^2 \le a_4[d(z, Bu)]^2$.

Hence $Bu = z$ i.e., $z = Tu = Bu$.

By condition (5), we have

$$d\left(TBu, BTu\right) = 0.$$

Hence $d(Tz, Bz) = 0$ i.e., $Tz = Bz$.

Now,

$$\left[d\left(z,Tz\right)\right]^2 = \left[d\left(Az,Bz\right)\right]^2 \le a_1 d\left(Az,Sz\right)d\left(Bz,Tz\right) + a_2 d\left(Bz,Sz\right)d\left(Az,Tz\right) + a_3 d\left(Az,Sz\right)d\left(Az,Tz\right) + a_4 d\left(Bz,Tz\right)d\left(Bz,Sz\right) + a_5 d^2\left(Sz,Tz\right)$$

i.e., $[d(z, Tz)]^2 \le a_2[d(z, Tz)]^2$ which is a contradiction. Hence $z = Tz$ i.e, $z = Tz = Bz$.

Therefore z is common fixed point of A, B, S and T. Similarly we can prove that z is a common fixed point of A, B, S and T if any one of A, B or T is continuous.

Finally, in order to prove the uniqueness of z, suppose w be another common fixed point of A, B, S and T Then we have,

$$\left[d\left(z,w\right)\right]^2 = \left[d\left(Az,Bw\right)\right]^2 \le a_1 d\left(Az,Sz\right)d\left(Bw,Tw\right) + a_2 d\left(Bw,Sz\right)d\left(Az,Tw\right) + a_3 d\left(Az,Sz\right)d\left(Az,Tw\right) + a_4 d\left(Bw\ Tw\right)d\left(Bw\ Sz\right) + a_5 d^2\left(Sz,Tw\right)$$

which gives $[d(z, Tw)]^2 \le a_2\,[d(z, Tw)]^2$. Hence $z = w$.

This completes the proof.

THEOREM 3.2: Let A, B, S and T be self maps of a complete metric space (X, d) satisfying the following conditions:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

(2) $[d(Ax, Bx)]^2 \le a_1d(Ax, Sx)d(By, Ty)+a_2d(By, Sx)d(Ax, Ty)+a_3d(Ax, Sx)d(Ax, Ty) +a_4d(By, Ty)d(By, Sx) +a_5d^2(Sx, Ty)$

where $a_1+ a_2 +2a_3 +a_4+ a_5< 1$ and $a_1, a_2, a_3, a_4, a_5 \ge 0$.

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on. Continuing this process we can define a sequence $\{y_n\}$ in X such that

$$y_{2n+1} = Tx_{2n+1} = Ax_{2n} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1}$$

then the sequence $\{y_n\}$ is Cauchy sequence in X.

(4) One of A, B, S or T is continuous.

(5) $[A, S]$ and $[B, T]$ are A-compatible mappings on X.

Then A, B, S and T have a unique common fixed point in X.

Proof: Similar to theorem 3.1.

Remark:

(i) By taking $a_1= a_2 =k_1$ and $a_3= a_4 =k_2$ and $a_5=0$ and (A, S) and (B, T) as compatible mappings theorem 3.1 reduces to theorem 1 of Bijendra and Chouhan [1].

(ii) By taking $S = T$ and (A, S) and (A, T) as commuting mappings or compatible mappings of type (A) theorem 3.1 reduce to results of Murthy [6] and Sharma and Sahu [8] under certain conditions.

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