

Periodic Asymptotic Homogenisation of a Clayey Sand–Kenaf Composite

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Abstract Adobe is a building material with high thermal inertia, but it is characterised by low mechanical strength, whether in compression or flexure, due to poor water resistance. Modern stabilisation techniques make it possible to significantly increase the ductility of these adobe units. In the literature, the improvement of the mechanical properties of adobe bricks through the incorporation of fibres, such as kenaf, has been experimentally demonstrated. Sampling-based analysis is extremely time-consuming and costly. A numerical simulation approach enables the optimal selection of composite adobe components to achieve superior ductility. This article presents a numerical study on the periodic asymptotic homogenisation of a clayey sand/kenaf composite intended for the manufacture of load-bearing earth bricks. The results reveal two distinct regimes: a quasi-isotropic dilute regime for volume fractions (V_f) $\leq 15\%$, and a concentrated regime with marked anisotropy for $V_f > 15\%$. The shear modulus G_{xy} is the most sensitive to the dosage and is quasi-proportional to the Young's modulus of the fiber. The local von Mises stress reaches its absolute maximum at $V_f = 20\%$, marking the upper limit of the recommended practical range, which is $[7 - 20\%]$ (0.58–1.91 wt.%). Finally, our model's predictions are validated by comparison with other methods and experimental results from the literature.

Keywords: plant-based composites, asymptotic homogenisation, finite element method, effective stiffness tensor

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1. Introduction

Facing a climate emergency, the construction sector is called upon to drastically reduce its CO₂ emissions: it alone accounts for nearly 39% of global greenhouse gas emissions [1]. Bio-based materials derived from renewable plant-based resources represent a promising response to this challenge: low embodied energy, carbon-neutral or even carbon-negative balance, and locally available resources [2,3]. Raw earth, used for millennia as a building material, offers major ecological and socio-economic advantages: abundance, low cost, natural hygrothermal regulation, and implementation accessible without industrial equipment [4,5]. More than 30% of the world population currently lives in earth constructions, a proportion exceeding 50% in some developing countries [4].

The main limitations of raw earth bricks (adobe) are their intrinsic brittleness (low tensile and shear strength) and water sensitivity. The incorporation of short plant fibres into the earth matrix is an ancient and effective technique to address these weaknesses, as it improves ductility, limits shrinkage cracking, and increases resistance to dynamic loading [4]. Recent studies have shown significant compressive and flexural strength gains

through the addition of sisal fibres [6], jute [7], coir (coconut fibre) [8], and kenaf [9,10]. Kenaf (*Hibiscus cannabinus*), a fast-growing plant widespread in the Sahel and West Africa, is particularly promising: its fibres exhibit tensile strength of 1 ± 0.25 GPa and a high elastic modulus ($E \approx 50$ GPa for a fibre bundle) [9,10,11], making it one of the stiffest plant reinforcements locally available.

Adding plant fibres to a clay matrix simultaneously improves compressive strength, flexural strength, and post-cracking ductility, while reducing thermal conductivity [9,12,13]. These gains, achieved at low mass dosages, result from two complementary mechanisms: crack bridging, which delays failure, and the reduction of macroporosity, which homogenises the microstructure [14,15].

The literature on fibre-reinforced bricks is essentially experimental: most studies characterise macroscopic properties (compressive, flexural and impact strength) of brick specimens as a function of fibre dosage and length [4,6,7,8]. Sanchez et al. [16] measured the mechanical properties of adobe masonry for building rehabilitation; Millogo et al. [9] characterised the properties of clayey sand from Burkina Faso and its kenaf fibres, quantifying the compressive strength gain. However, few works address multi-scale numerical modelling of these composites: the relationship between microstructural

properties (volume fraction, inclusion geometry, matrix/fibre stiffness contrast) and the effective properties of the brick remains poorly elucidated. Most existing models use phenomenological approaches or Voigt/Reuss mixing rules that do not capture microstructural effects [17]. Research on kenaf-reinforced composites is attracting increasing attention owing to their excellent properties and their favourable ecological profile. Building on these findings, the present study investigates bricks composed of kenaf fibres and clayey sand.

In standard homogenisation techniques for fibre-matrix composites, the fibres are assumed to have uniform circular cross-sections, allowing for a single-fibre representative volume element (RVE) approach. This assumption is not applicable to plant fibres, such as kenaf, which exhibit variations in radius and length. Our approach introduces a novel modelling method based on integrating fibre bundles that correspond to the total fibre content within the volume. Consequently, the analysis shifts from considering individual fibre counts to the quantity of fibres represented by the necessary fibre bundles. This modelling strategy, inspired by the single-inclusion periodic-cell concept of [18], is motivated by research demonstrating that periodic and random microstructures yield equivalent results at moderate volume fractions [19,20]. It constitutes an original contribution of the present work.

The periodic asymptotic homogenisation method (PAH) provides a rigorous mathematical framework for bridging the microstructural and macrostructural scales [18,21]. Unlike Mori-Tanaka or mixing-law approaches, PAH provides: (i) the exact effective stiffness tensor for a given periodic microstructure, and (ii) local stress and strain fields within the microstructure (localisation), enabling identification of micro-cracking risk zones [18,22]. The method has been successfully applied to synthetic fibre composites (CFRP [23], short fibres [24,25]) and to biological composites (cellulosic fibre networks [26]). Its application to earth/plant-fibre composites remains largely unexplored to date [2,4].

Multi-scale modelling of these composites requires an explicit representation of the microstructure at the pattern scale ($\sim 20\text{mm}$), combined with rigorous homogenisation at the brick scale ($\sim 200\text{mm}$). The quality of periodic boundary conditions (PBC) is critical [22,27]: inaccurate PBCs (unmatched nodes at opposite boundaries) lead to null or erroneous results. Several approaches exist for numerical PAH implementation [24,25]: commercial software (ABAQUS, COMSOL) or dedicated codes; the present study adopts a dedicated Python code coupled to the Gmsh API [28], ensuring complete control of the pipeline (meshing, assembly, solving, localisation).

Previous works on this composite considered fixed volume fractions and regular meshes. The present paper proposes an original approach that:

- introduces the number of elementary fibres as a physically grounded control parameter, from which the bundle radius and volume fraction are analytically derived; varying the fibre density

within the brick thus corresponds to varying the bundle size within the reference volume;

- uses a Gmsh globally conforming unstructured mesh ensuring exact PBCs by construction;
- analyses 15 volume fractions ranging from 1% to 40%

The aim of this article is to predict the mechanical parameters or the effective stiffness tensor of sandy clay/kenaf composites in order to assist manufacturing engineers in the brick-production process.

To this end, this article first introduces the definition of the homogenisation problem for a periodic medium, then establishes the relationship between the elasticity tensor, the shear modulus, and the 2D elasticity coefficients calculated on the Representative Volume Element (RVE). We then present the associated finite element numerical experiment, followed by a critical analysis and in-depth discussion, before finally concluding the study with a summary and an outlook for future research.

2. Linear Elasticity Problem Modelling via Periodic Asymptotic Homogenisation

Asymptotic homogenisation is a rigorous mathematical method for determining the effective behaviour of a periodic heterogeneous material from the knowledge of its microstructure. It was independently developed in the 1970s–1980s by Bensoussan et al [29], then by Sanchez-Palencia [30], and formalised from a finite element standpoint by Guedes et al [31].

2.1. Problem Definition

The central idea rests on the introduction of a small parameter ϵ representing the ratio between the characteristic size of the microstructure l and that of the macroscopic structure L .

The ratio between the characteristic dimension of the pattern and that of the brick defines the parameter ϵ :

$$\epsilon = \frac{l}{L} = \frac{20}{220} \approx 0.091 \ll 1 \quad (1)$$

The condition $\epsilon \ll 1$ justifies the use of the periodic asymptotic homogenisation method (PAH) [18,21]. When $\epsilon \rightarrow 0$ (Figure 1), the displacement field u^ϵ converges (in the weak H^1 sense) towards a macroscopic field solution of an elasticity problem with a homogenised tensor $\mathcal{C}^{\text{hom}}$ [32,33].

This approach has been successfully applied to fibre composites [18,21], porous materials [34], lattice structures [35], and more recently to materials combining cementitious binders with natural reinforcements [36]. We assumed the kenaf fibre/clayey sand mixture to be sufficiently homogeneous to justify treating the composite brick as a homogeneous medium for the purposes of the present 2D study.

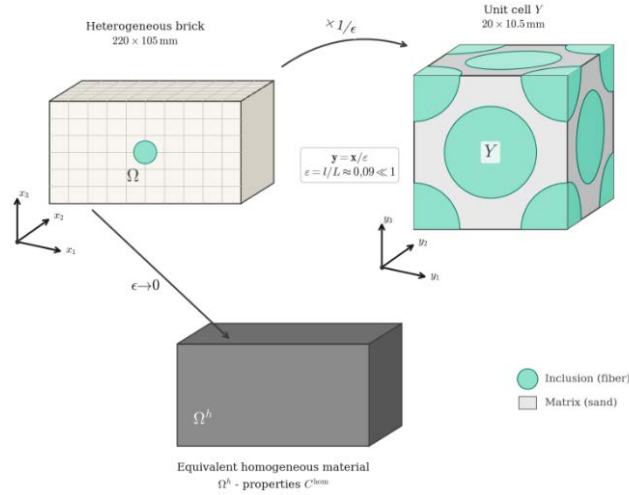


Figure 1. Principle of periodic asymptotic homogenisation: transition from the heterogeneous domain Ω to the unit cell Y , then to the equivalent homogeneous material ($\epsilon \rightarrow 0$)

Two coordinate systems are defined in the plane, under the plane-stress assumption:

- $x = (x_1, x_2) \in \mathbb{R}^2$: macroscopic coordinates,

associated with the global domain $\Omega \subset \mathbb{R}^2$ with characteristic dimensions L ; they describe the position of a point within the equivalent homogeneous structure;

- $y = x / \epsilon = (y_1, y_2) \in Y$: microscopic coordinates,

associated with the unit cell $Y \subset \mathbb{R}^2$ with characteristic dimensions l ; they describe the position of a point within the periodic microstructure.

The elasticity tensor is Y -periodic:

$$C_{ijkl}^\epsilon(x) = C_{ijkl}\left(\frac{x}{\epsilon}\right) = C_{ijkl}(y) \quad (2)$$

Assuming small deformations and static equilibrium, the linear elasticity problem reads [18]:

$$\frac{\partial \sigma_{ij}^\epsilon}{\partial x_j} + f_i = 0 \text{ in } \Omega \rightarrow (3)$$

$$\epsilon_{ij}^\epsilon = \frac{1}{2} \left(\frac{\partial u_i^\epsilon}{\partial x_j} + \frac{\partial u_j^\epsilon}{\partial x_i} \right) \text{ in } \Omega \rightarrow (4)$$

$$\sigma_{ij}^\epsilon = C_{ijkl}^\epsilon \epsilon_{kl}^\epsilon \text{ in } \Omega \rightarrow (5)$$

with boundary conditions:

$$u_i^\epsilon = \bar{u}_i \text{ on } \Gamma_u, \quad \sigma_{ij}^\epsilon n_j = \bar{t}_i \text{ on } \Gamma_F \quad (6)$$

where σ_{ij}^ϵ and ϵ_{ij}^ϵ are the components of the Cauchy stress and strain tensors; f_i and u_i^ϵ are the components of the body force and displacement fields; $\bar{u}_i$ and $\bar{t}_i$ are the prescribed displacements and surface tractions on boundaries Γ_u and Γ_F of Ω , with $\Gamma_u \cup \Gamma_F = \Gamma$ and $\Gamma_u \cap \Gamma_F = \emptyset$; n_j is the outward normal to Γ_F (see Figure 2).

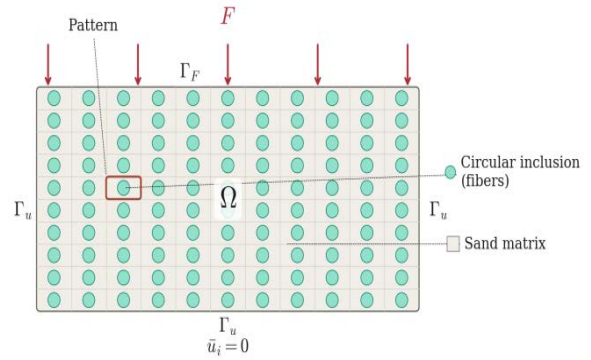


Figure 2. Problem definition

2.2. Periodic Asymptotic Formulation

The periodicity of the microstructure introduces two coupled coordinate systems: x (macroscopic, brick scale Ω) and $y = x / \epsilon$ (microscopic, cell scale Y), related by [18]:

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} + \frac{1}{\epsilon} \frac{\partial}{\partial y_i} \quad (7)$$

The elasticity tensor $C_{ijkl}(y)$ is Y -periodic in y : its components do not depend on the macroscopic coordinates x , but oscillate rapidly at the microstructure scale ϵ [18,21].

The displacement solution field $u_i^\epsilon(x)$ is approximated by an asymptotic expansion in ϵ [18]:

$$u_i^\epsilon(x) = u_i^{(0)}(x, y) + \epsilon u_i^{(1)}(x, y) + \epsilon^2 u_i^{(2)}(x, y) + \dots \quad (8)$$

where the terms $u_i^{(r)}(x, y)$, $r \in \mathbb{N}$, are Y -periodic functions called order- r correctors. The strain and stress tensors admit similar expansions obtained by substituting (8) into relations (4) and (5) and using rule (7):

$$e_{ij}^\epsilon = \epsilon^{-1} e_{ij}^{(0)} + \epsilon^0 e_{ij}^{(1)} + \epsilon^1 e_{ij}^{(2)} + \dots (9)$$

$$\sigma_{ij}^\epsilon = \epsilon^{-1} \sigma_{ij}^{(0)} + \epsilon^0 \sigma_{ij}^{(1)} + \epsilon^1 \sigma_{ij}^{(2)} + \dots (10)$$

Substituting (10) into the equilibrium equation (3) and grouping by powers of ϵ , a hierarchy of differential equations is obtained [18]:

$$\epsilon^{-2} : \quad \frac{\partial \sigma_{ij}^{(0)}}{\partial y_j} = 0 \rightarrow (11)$$

$$\epsilon^{-1} : \quad \frac{\partial \sigma_{ij}^{(0)}}{\partial x_j} + \frac{\partial \sigma_{ij}^{(1)}}{\partial y_j} = 0 \rightarrow (12)$$

$$\epsilon^0 : \quad \frac{\partial \sigma_{ij}^{(1)}}{\partial x_j} + \frac{\partial \sigma_{ij}^{(2)}}{\partial y_j} + f_i = 0 \rightarrow (13)$$

From the ϵ^{-2} -order equation (11) and the constitutive law, one deduces that $\sigma_{ij}^{(0)}$ is independent of y , which implies [18]:

$$\sigma_{ij}^{(0)} = 0 \text{ and } u_i^{(0)} = u_i^{(0)}(x) \quad (14)$$

In other words,

- $u_i^{(0)}$ is independent of the microscopic coordinates: $u_i^{(0)} = u_i^{(0)}(x)$; it is the homogenised macroscopic displacement field;
- $u_i^{(1)}$ is the first-order corrector; it represents the microscopic fluctuations superimposed on the macroscopic field.

$$u_i^{(1)}(x, y) = -\chi_i^{mn}(y) \frac{\partial u_m^{(0)}}{\partial x_n}(x) \quad (15)$$

where $\chi_i^{mn}(y)$ is the corrector field, components of the first-order displacement tensor, Y -periodic and zero-mean over Y .

We define the space of admissible periodic virtual displacements as:

$$V_Y^\epsilon = \left\{ v \in H^1(Y)^2 \mid v \text{ is } Y\text{-periodic} \right\} \quad (16)$$

where $H^1(Y)^2$ denotes the Sobolev space of displacement fields taking values in $\mathbb{R}^2$ that are square-integrable, along with their first derivatives, over Y [29,32]:

$$H_2^1(Y) = \left\{ v : Y \rightarrow \mathbb{R}^2 \mid \int_Y |v|^2 dy < \infty, \int_Y |\nabla v|^2 dy < \infty \right\} \quad (17)$$

The Y -periodicity condition imposes:

$$v(y + k e_i) = v(y) \quad \forall y \in \partial Y, k \in \mathbb{Z}, i = 1, 2 \quad (18)$$

where e_i denotes the i -th vector of the canonical basis of $\mathbb{R}^2$ and $k \in \mathbb{Z}$ is an integer representing the number of

translation periods [29,32]. That is, v takes identical values on opposite boundaries of Y . Substituting (15) into the ϵ^{-1} order equation (12), the components $\chi_i^{mn} \in V_Y^\epsilon$ are solutions of the variational problem [18]:

$$\int_Y C_{ijkl} \frac{\partial \chi_k^{mn}}{\partial y_l} \frac{\partial v_i}{\partial y_j} dY = \int_Y C_{ijmn} \frac{\partial v_i}{\partial y_j} dY \quad \forall v \in V_Y^\epsilon \quad (18)$$

In plane stress, three independent linear systems are solved: $mn \in \{11, 22, 12\}$; their solutions constitute the correctors χ^{11} , χ^{22} and χ^{12} . Once the correctors χ^{mn} are computed, the homogenised stiffness tensor C^{hom} is obtained by integration over the unit cell Y [18]:

$$C_{mnpq}^{hom} = \frac{1}{|Y|} \int_Y C_{ijpq}(y) \left(\delta_{im} \delta_{jn} - \frac{\partial \chi_i^{mn}}{\partial y_j} \right) dY \quad (20)$$

C_{mnpq}^{hom} contains complete information on the effective mechanical behaviour: it is constant over the entire macroscopic domain Ω and depends only on the geometry of cell Y and local properties $C_{ijkl}(y)$. The macroscopic displacement field $u_i^{(0)}(x)$ is then solution of the homogenised elasticity problem [18]:

$$\frac{\partial}{\partial x_j} \left(C_{ijkl}^{hom} \frac{\partial u_k^{(0)}}{\partial x_l} \right) + f_i = 0 \text{ in } \Omega \quad (21)$$

in which the original heterogeneous material has been substituted by a homogeneous equivalent material of stiffness C^{hom} . The determination of the macroscopic displacement field corresponds to the solution $u^{(0)} \in V_\Omega^0$ of the variational problem

$$\int_\Omega C_{ijkl}^{hom} \frac{\partial u_k^{(0)}}{\partial x_l} \frac{\partial v_i}{\partial y_j} d\Omega = \int_\Omega f_i v_i d\Omega + \int_\Gamma \tilde{t}_i v_i d\Gamma, \quad \forall v \in V_\Omega^0 \quad (22)$$

where V_Ω^0 is the set of continuous and sufficiently regular functions, zero-valued in Γ_u .

The numerical resolution of (22) is thus independent of ϵ : it does not see microstructural details, considerably reducing the number of degrees of freedom compared to the original problem (3) [18,21].

For $\epsilon \ll 1$, higher-order correctors are negligible compared to the first-order term [18]. The conventional approximation retains only:

$$u_i^\epsilon(x) \approx u_i^{(0)}(x) + \epsilon \chi_i^{mn}(y) \frac{\partial u_m^{(0)}}{\partial x_n}(x) \quad (23)$$

The solution $u_i^{(0)}(x)$ is obtained by numerical resolution of problem (22) over the macroscopic domain Ω .

2.3. Finite Element Discretisation

The numerical implementation of the PAH follows the formulation of [18,21]. The mesh consists of three-node triangular elements (T3), also known as P1 Lagrange elements. Each triangle e is defined by its three nodes $\mathbf{x}_e^{(1)}$, $\mathbf{x}_e^{(2)}$, $\mathbf{x}_e^{(3)}$ and its six nodal degrees of freedom. For each triangular element T3 of the mesh, the deformation field is linked to the nodal displacements by the kinematic matrix $\mathbf{B}_e \in \mathbb{R}^{3 \times 6}$, of partial derivatives of the interpolation functions:

$$\varepsilon_e = \mathbf{B}_e \mathbf{u}_e, \quad \sigma_e = \mathbf{C}_e \mathbf{B}_e \mathbf{u}_e \quad (24)$$

where $\varepsilon_e \in \mathbb{R}^3$ is the deformations vector of the element e in Voigt notation $(\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12})^T$, $\mathbf{u}_e \in \mathbb{R}^6$ the vector of nodal displacements, and $\mathbf{C}_e \in \mathbb{R}^{3 \times 3}$ the tensor of rigidity of the phase of the element e (kenaf or clayey sand) in plane constraints [37,38].

The discretisation of the problem of cell (19) leads to a global linear system:

$$\left(\sum_e \int_{Y_e} \mathbf{B}_e^T \mathbf{C}_e \mathbf{B}_e dY \right) \chi^{mn} = \sum_e \int_{Y_e} \mathbf{B}_e^T \mathbf{C}_e dY \mathbf{F}^{mn} \quad (25)$$

where $\mathbf{K}_{per} \in \mathbb{R}^{N \times N}$ is the periodic global stiffness matrix, N the number of free degrees of freedom after condensation, $\chi^{mn} \in \mathbb{R}^N$ the vector of nodal correctors of the mn mode, and $\mathbf{F}^{mn} \in \mathbb{R}^N$ the unit loading vector associated with the mode $mn \in \{11, 22, 12\}$ [21,29]. Periodic boundary conditions are imposed via slave/master condensation on pairs of matched nodes at opposite edges:

$$\chi^{mn}(\mathbf{x} + \mathbf{Y}) = \chi^{mn}(\mathbf{x}) \quad \forall \mathbf{x} \in \partial Y$$

Once the three systems (25) are solved, the tensor $\mathbf{C}^{hom}$ is obtained by quadrature over the cell [18,31]:

$$\mathbf{C}_{mnpq}^{hom} = \frac{1}{|Y|} \sum_{e=1}^E |Y_e| \mathbf{C}_{ijpq}^{(e)} \left(\delta_{im} \delta_{jn} - \mathbf{B}_{ij,k}^{(e)} \chi_{e,k}^{mn} \right) \quad (27)$$

where $|Y_e|$ is the area of element e , $|Y|$ is the total area of the cell, δ_{im} is the Kronecker delta, and $\chi_{e,k}^{mn}$ is the k -th component of the corrector in element e [18]. The microstructural strain and stress fields (discrete localization) are reconstructed by [31]:

$$\varepsilon_e^{(1)} = (I - \mathbf{B}_e \chi_e) \bar{\varepsilon} \rightarrow (28)$$

$$\sigma_e^{(1)} = \mathbf{C}_e (I - \mathbf{B}_e \chi_e) \bar{\varepsilon} \rightarrow (29)$$

where $\bar{\varepsilon}$ is the imposed macroscopic strain and I is the 3×3 identity matrix. These fields allow for the calculation of the local Von Mises stress σ_{VM} at the

kenaf/clayey sand interface, an indicator of the risk of micro-cracking [18,22].

The three systems (25) ($mn \in \{11, 22, 12\}$) are solved using a sparse direct solver applied to the sparse matrices generated during assembly; the sparse structure of $\mathbf{K}_{per}$ (local connectivity of T3 elements) is exploited to minimise memory usage and computation time [18]. The complete numerical code is implemented in Python 3 and relies on the Gmsh Python API [28] for generating conforming periodic meshes. This API provides full programmatic access to the meshing engine without requiring the graphical user interface: geometry, Boolean operations, and refinement fields are defined entirely through Python function calls.

3. Relation between the Elasticity Tensor, Shear Modulus and Elastic Coefficients in 2D

In continuum mechanics, the elastic behaviour of a material is fully described by its stiffness tensor $\mathbf{C}$ [37]. For a heterogeneous composite, homogenisation yields the effective tensor $\mathbf{C}^{hom}$, which replaces the heterogeneous material with an equivalent homogeneous medium [18,29]. This section establishes the relationship between $\mathbf{C}^{hom}$ and the classical engineering constants (E_x , E_y , G_{xy} , ν_{yx}), which serve as the parameters directly used for structural design.

3.1. Orthotropic form of $\mathbf{C}^{hom}$ under Plane Stress

Under plane stress, the homogenised stiffness tensor $\mathbf{C}^{hom}$ is a 3×3 matrix in Voigt notation ($\{11, 22, 12\}$) [37,38]. For a composite with two axes of symmetry, which is the case for the rectangular cell with a centred circular inclusion, anisotropic elasticity theory predicts the orthotropic form [18,22]:

$$\mathbf{C}^{hom} = \begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{pmatrix} \quad (30)$$

where:

- C_{11} and C_{22} are the directional stiffness moduli along x and y ;
- C_{12} is the coupling term between tension and lateral contraction;
- C_{33} is the shear stiffness modulus in the xy - plane;
- $C_{13} = C_{23} = 0$ due to orthotropic symmetry.

These off-diagonal terms would couple the normal and shear modes; such coupling is precluded by the two-fold symmetry of the microstructure [37]. The Maxwell-Betti symmetry relationship dictates [38]:

$$\frac{\nu_{yx}}{E_x} = \frac{\nu_{xy}}{E_y} \quad (31)$$

which reduces the number of independent moduli under plane stress conditions to four: E_x , E_y , G_{xy} , and ν_{yx} (or ν_{xy} , as one is derived from the other via (31)).

3.2. Analytical Inversion: Components of C^{hom} to Engineering Modulus

The flexibility tensor $S^{\text{hom}} = (C^{\text{hom}})^{-1}$ is obtained by analytical inversion from the block 2×2 [37,38]:

$$S^{\text{hom}} = \begin{pmatrix} 1/E_x & -\nu_{xy}/E_y & 0 \\ -\nu_{yx}/E_x & 1/E_y & 0 \\ 0 & 0 & 1/G_{xy} \end{pmatrix} \quad (32)$$

The inversion of the 2×2 block of C^{hom} (30) leads to explicit relationships [18]:

$$E_x = \frac{\det C}{C_{22}}, E_y = \frac{\det C}{C_{11}} \rightarrow (33)$$

$$G_{xy} = C_{33} \rightarrow (34)$$

$$\nu_{yx} = \frac{C_{12}}{C_{11}}, \nu_{xy} = \frac{C_{12}}{C_{22}} \rightarrow (35)$$

where the determinant of the block 2×2 is:

$$\det C = C_{11} C_{22} - C_{12}^2 \quad (36)$$

Physical significance.

- $E_y = \det C / C_{11}$: the transverse modulus is inversely proportional to C_{11} ; in the concentrated regime, $C_{22} \rightarrow \infty$ leads to $\det C \rightarrow \infty$, hence the explosive increase in E_y at high volume fractions [18,19].

- $G_{xy} = C_{33}$: shear behaviour is governed entirely by the single component C_{33} , without coupling to normal modes (a consequence of orthotropy) [37,38].

- $\nu_{yx} = C_{12} / C_{11}$: the Poisson's ratio depends on the ratio C_{12} / C_{11} ; the quasi-invariance of C_{12} observed in this study implies that ν_{yx} decreases primarily due to the increase in C_{11} with V_f [18,29,38].

3.3. Independence of Moduli and Information Hierarchy

The four independent moduli do not contain the same information regarding the microstructure:

E_y governed by geometric confinement (C_{22}); virtually insensitive to E_1 in the concentrated regime [19].

G_{xy} practically proportional to E_1 ; the only modulus that allows for distinguishing between fibre choices [22,27].

C_{12} exclusive signature of the matrix; independent of E_1 and V_f : an original result of the present study, consistent with the theoretical prediction of Bensoussan et al. [29].

ν_{yx} decreases steadily with V_f ; derived from C_{12} and C_{11} via (35) [38].

This information hierarchy justifies focusing the comparative study of fibres on E_y , G_{xy} , and C_{12} . These are the three quantities that contain the most information regarding the effect of fibre choice and volume fraction.

4. Numerical Experimentation

This section implements the numerical experimentation on the representative volume element (RVE) to validate the theoretical formulation, using the Gmsh API for meshing and the finite element method (FEM) for resolution. Periodic boundary conditions are incorporated to solve the local problems and extract the localisation fields required for computing the homogenised properties.

4.1. Material Properties

The matrix consists of clayey sand classified as USCS SC [39], with a Young's modulus $E_m = 30$ MPa and a Poisson's ratio $\nu_m = 0.35$. This value corresponds to the midpoint of the $[10;50]$ MPa range for loose clayey sands [40,41,42]. The reinforcement is an equivalent bundle of kenaf fibres with a Young's modulus $E_f = 50$ GPa and a Poisson's ratio $\nu_f = 0.33$ [9,10,11]. The stiffness contrast ratio $E_f / E_m \approx 1667$ justifies the PAH approach [18].

4.2. Experimental Setup

The modelled brick is a rectangle with dimensions $L \times l = 220 \times 105$ mm, which are standard dimensions for compressed earth bricks used in Togo [12,43]. It is composed of $N \times M = 11 \times 10 = 110$ identical patterns or unit cells with dimensions $L_m \times H_m = 20 \times 10.5$ mm (Fig. 2). The pattern is the unit cell Y of the periodic medium; it is the smallest geometric unit that, through periodic repetition in both directions, reconstructs the complete brick. Each pattern contains a single circular inclusion which is the kenaf bundle with radius r_{eq} centered within a clayey sand matrix. This approach is defined by the following sequence: the radius of the elementary fibre $r_f = 34 \mu\text{m}$ [10] is determined by the plant biology; the number of elementary fibres N_f , the sole control parameter is defined and the equivalent bundle radius r_{eq} and the volume fraction V_f are then derived from this.

$$\underbrace{r_f = 34 \mu\text{m}}_{\text{fixed}} \xrightarrow{\text{choose } N_f} r_{eq} = r_f \sqrt{N_f} \text{ stems from } V_f \quad (37)$$

where K is the proportionality coefficient calculated analytically by:

$$K = \frac{A_{pat}}{A_{fib}} = 57,824 \quad (38)$$

where $A_{pat} = L_m \times H_m$ is the area of the pattern and $A_{fib} = \pi r_f^2$ is the cross-sectional area of the fibre.

When the equivalent beam radius approaches half the height of the pattern, either $r_{eq} \rightarrow H_m / 2 = 5.25 \text{ mm}$, the inclusion spans the entire height of the pattern and comes into contact with the top and bottom edges. This configuration corresponds to the model's geometric packing limit:

$$\begin{aligned} V_f^{\max} &= \frac{\pi (H_m / 2)^2}{A_{pat}} = \frac{\pi \times 5.25^2}{210} \\ &= 41.23\% \Rightarrow N_f^{\max} = 23,843 \end{aligned} \quad (39)$$

Beyond this value, inclusions from adjacent patterns geometrically overlap: periodicity is no longer maintained, and the model loses its physical validity.

The Gmsh computational domain covers the entire brick $220 \times 105 \text{ mm}$ to ensure mesh conformity at the boundaries for periodic matching; however, the effective properties C^{hom} are those of the pattern Y , independent of the brick size [18,29].

While the number of individual fibres N_f and consequently the volume fraction V_f , govern the theoretical formulation of our approach, for the practitioner, the quantity that can be measured during mixing is the mass of fibres added to a given amount of clayey sand. It is therefore necessary to establish the correspondence between the two formulations in order to make the model results directly applicable on site. Thus, the relationship between the volume fraction V_f and the mass fraction w_f is established based on the densities of the two phases:

$$w_f = \frac{m_{kenaf}}{m_{kenaf} + m_{clayey\ sand}} = \frac{\rho_1 V_f}{\rho_1 V_f + \rho_2 (1 - V_f)} \quad (40)$$

and vice vers:

$$V_f = \frac{\rho_2 w_f}{\rho_2 w_f + \rho_1 (1 - w_f)} \quad (41)$$

where

m_{kenaf} and $m_{clayey\ sand}$ are the mass of the kenaf and the mass of the clayey sand, respectively;

$\rho_1 \in [0.14; 0.15] \text{ g/cm}^3$ [44] is the apparent density of bulk kenaf;

$\rho_2 \in [1.4; 1.8] \text{ g/cm}^3$ [40,41] is the bulk density of the compacted clayey sand.

By adopting $\rho_1 = 0.14 \text{ g/cm}^3$ and $\rho_2 = 1.8 \text{ g/cm}^3$ (most frequently cited value for compacted clayey sand),

the relationship between V_f , N_f , and w_f is given in Table 1 for the key fractions:

$$w_f = \frac{7V_f}{90 - 83V_f} \quad (42)$$

Table 1. Relationship between volume fraction V_f , number of elementary fibres N_f , and mass fraction w_f (kenaf/composite). $\rho_1 = 0.14 \text{ g/cm}^3$ [44]; $\rho_2 = 1.8 \text{ g/cm}^3$ [41]

V_f (%)	N_f	w_f (%)	Interpretation
1	579	0.08	Minimum dosage
3	1,735	0.24	Low practical dosage
7	4,048	0.58	Recommended lower limit
12	6,939	1.05	Recommended range
20	11,565	1.91	Recommended upper limit
40	23,130	4.93	Practical limit of the model

5. Results

5.1. Microscopic Fields

The localisation procedure constitutes the inverse operation of homogenisation: knowing the macroscopic displacement field $u_i^{(0)}(x)$ and the correctors $\chi_i^{mn}(y)$, it allows reconstructing the strain and stress fields inside the microstructure, cell Y scale at any point x of Ω [18,21].

In plane stress, the discrete approximations of microscopic fields for each element e of the mesh read:

$$\varepsilon_e^{(1)}(x, y) = (I - B_e \chi_e) \varepsilon^{(0)}(x) \rightarrow (43)$$

$$\sigma_e^{(1)}(x, y) = C_e (I - B_e \chi_e) \varepsilon^{(0)}(x) \rightarrow (44)$$

where B_e is the kinematic matrix of element T3, C_e its elasticity tensor, χ_e the corrector value at the element nodes, and $\varepsilon^{(0)}$ the homogenised macroscopic strain. Physically, the term $(I - B_e \chi_e)$ represents the localisation tensor: it redistributes the macroscopic strain between the kenaf and clayey sand phases according to their respective stiffnesses.

To assess the risk of local plastification or micro-cracking, the Von Mises stress is computed from fields (44):

$$\sigma_{VM} = \sqrt{\left(\sigma_{11}^{(1)}\right)^2 - \sigma_{11}^{(1)} \sigma_{22}^{(1)} + \left(\sigma_{22}^{(1)}\right)^2 + 3\left(\sigma_{12}^{(1)}\right)^2}$$

5.2. Homogenised Stiffness Tensor

Figures 3 presents the four independent components of the homogenised stiffness tensor C^{hom} under plane stress conditions for the 15 volume fractions. The off-diagonal coupling terms satisfy $\max(|C_{13}|, |C_{23}|) < 1.3 \times 10^{-2} \text{ MPa}$

over the entire range, i.e less than 7×10^{-4} times C_{33} . These residual values represent the numerical noise of the sparse solver and confirm the orthotropic symmetry predicted by the two-axis symmetry geometry of the

pattern [18,22]. The Voigt form(30) is thus numerically validated.

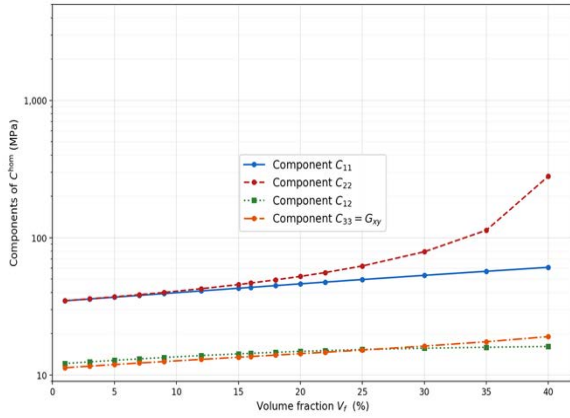


Figure 3. Components of C^{hom} : Asymptotic homogenisation

5.3. Engineering Moduli and Two Behavioural Regimes

The engineering moduli are derived from the homogenised stiffness tensor C^{hom} by analytically inverting the 2×2 block (33-36). Figures 3 and 4 (coefficient ν_{yx} , components C_{11} and C_{22}) illustrate the evolution of these moduli as a function of volume fraction. The results reveal two distinct regimes separated at $V_f \approx 15\%$, which is consistent with the transition observed in unidirectional composites between the dilute and concentrated inclusion behaviors [19,45]. The coefficient ν_{yx} , calculated using the PAH and Eshelby-based techniques [45], is compared and found to decrease with V_f (Figure 4). This is consistent with the results of Penta et al. [46].

The Maxwell–Betti reciprocity relation is verified numerically [38]:

$$\frac{\nu_{yx}}{E_x} = \frac{\nu_{xy}}{E_y} = \frac{C_{12}}{\det C} \quad (46)$$

The ratio E_x / E_y constitutes an anisotropy index of the composite, which measures the difference in stiffness between the two principal directions and allows the two behavioural regimes to be defined. For volume fractions between 1% and 15% where $E_x / E_y > 0.94$, the regime is considered dilute; the brick exhibits quasi-isotropic behaviour and can be sized without regard to the installation direction. (see Figure5). Beyond 15% where the E_x / E_y ratio decreases from 0.93 at $V_f = 16\%$ to 0.22 at $V_f = 40\%$, the regime is termed concentrated and the brick exhibits anisotropic behaviour; consequently, the laying direction becomes a design parameter in its own right [37,38].

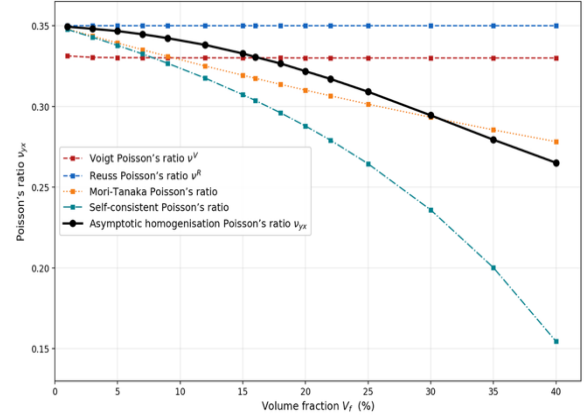


Figure 4. Poisson's ratio: Asymptotic homogenisation vs Eshelby based techniques

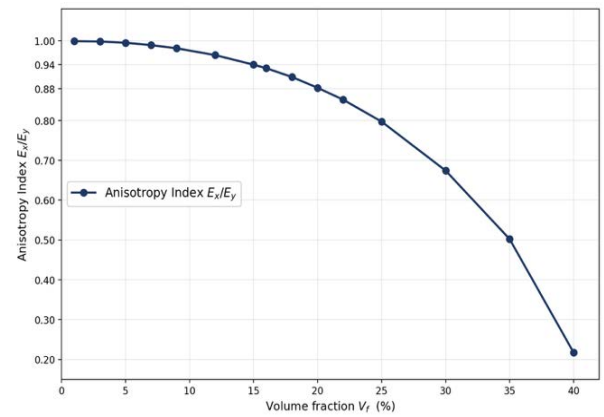


Figure 5. Anisotropy index E_x / E_y

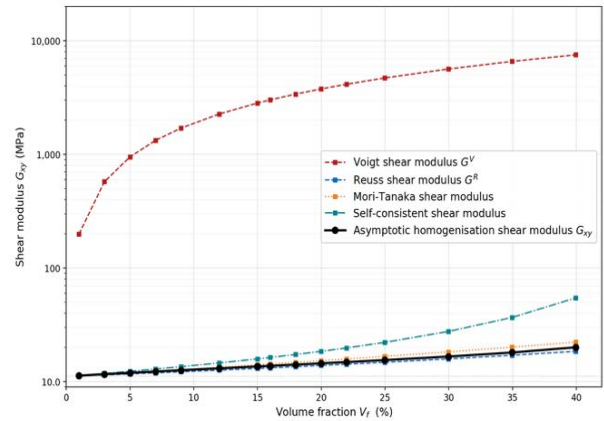


Figure 6. Shear modulus: Asymptotic homogenisation vs Eshelby based techniques

The reinforcement efficiency is quantified by the relative gain in the transverse Young's modulus:

$$\text{Gain}_{E_y} = \frac{E_y^{\text{hom}} - E_m}{E_m} \times 100\%, E_m = 30 \text{ MPa} \quad (47)$$

The shear modulus defined by relation (34) exhibits the lowest sensitivity to fibre content, with a ratio of only 1.78

between the 1% level where $G_{xy} = 11.3$ MPa and the 40% level where $G_{xy} = 20.1$ MPa (see Figure 6). In the same figure, the shear modulus G_{xy} is compared with shear moduli determined by other methods based on the Eshelby model, in accordance with the work of Penta et al. [46].

5.4. Validation using Voigt–Reuss bounds and Eshelby model

A necessary condition for any homogenisation result is the verification of the Voigt and Reuss bounds [29,47]:

$$E^{Reuss} \leq E_y^{hom} \leq E^{Voigt} \quad (48)$$

where, for a two-phase composite under plane stress:

$$E^{Voigt} = V_f E_1 + (1 - V_f) E_2 \rightarrow (49)$$

$$E^{Reuss} = \left(\frac{V_f}{E_1} + \frac{1 - V_f}{E_2} \right)^{-1} \rightarrow (50)$$

The Mori-Tanaka bulk [48,49,50] and shear moduli are expressed as follows:

$$K^{MT} = K_2 + \frac{V_f (K_1 - K_2)}{1 + (1 - V_f) \frac{K_1 - K_2}{K_2 + G_2}} \quad \text{with}$$

$$K_i = E_i / [2(1 - \nu_i)]$$

$$G^{MT} = G_2 + \frac{V_f (G_1 - G_2)}{1 + (1 - V_f) \frac{G_1 - G_2}{G^*}} \quad \text{where}$$

$$G^* = \frac{2G_2(K_2 + G_2)}{K_2 + 2G_2}$$

then the transverse Young's modulus and Poisson's ratio by:

$$E^{MT} = \frac{4K^{MT}G^{MT}}{K^{MT} + G^{MT}} \nu^{MT} = \frac{K^{MT} - G^{MT}}{K^{MT} + G^{MT}}$$

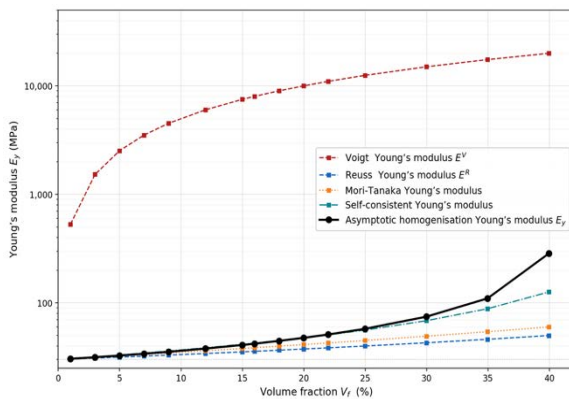


Figure 7. Young's modulus: Asymptotic homogenisation vs Eshelby-based techniques

Figure 7 verifies inequality (48) for the 15 volume fractions. The proximity of E_y^{hom} to the Reuss bound confirms a series-like behaviour; the compliant clayey sand matrix governs the overall response even at the

highest volume fractions, consistent with the matrix-dominated regime identified by Pinho-da-Cruz et al. [18] for composites with high stiffness contrast. The transverse Young's modulus is compared with Young's moduli calculated using other methods based on the Eshelby model, as done by Penta et al. [46].

6. Discussion

6.1. Anisotropy and Pattern Shape

6.1.1. Evolution of the Anisotropy Index

The ratio $E_x / E_y = C_{11} / C_{22}$ represents the composite's anisotropy index; it equals 1 for an in-plane isotropic material and decreases towards 0 as C_{22} dominates C_{11} . Over the range studied:

$$\frac{E_x}{E_y} = \frac{C_{11}}{C_{22}} : 0.999 \xrightarrow{V_f = 1\% \rightarrow 40\%} 0.217 \quad (53)$$

This decrease is not uniform. It is negligible in the dilute regime ($E_x / E_y > 0.94$ for $V_f \leq 15\%$) and accelerates sharply in the concentrated regime, as shown in Figure 7. The transition at $V_f \approx 15\%$ therefore also marks the threshold beyond which anisotropy becomes a design parameter that must be taken into account.

6.1.2. Geometric Origin of Anisotropy

This anisotropy stems directly from the rectangular shape of the unit cell, with dimensions $L_m = 20 \text{ mm} \neq H_m = 10.5 \text{ mm}$ such that $L_m / H_m = 1.905$. Indeed, for a given volume fraction, the radius of the equivalent beam $r_{eq} = r_f \sqrt{N_f}$ is the same in both directions; however, the residual matrix layer between the inclusion and the unit cell boundary is thinner in the H_m direction than in the L_m direction:

$$e_y = \frac{H_m}{2} - r_{eq} < e_x = \frac{L_m}{2} - r_{eq} \quad (54)$$

The H_m direction therefore saturates first. Consequently, C_{22} increases faster than C_{11} , resulting in $C_{22} \gg C_{11}$ at high volume fractions [18,19].

By way of illustration, at $V_f = 40\%$ where $r_{eq} = 5.171 \text{ mm}$:

$$e_y = 5.250 - 5.171 = 0.079 \text{ mm} \rightarrow (55)$$

$$e_x = 10.000 - 5.171 = 4.829 \text{ mm} \rightarrow (56)$$

This extreme ratio explains the drop in E_x / E_y to 0.217 at $V_f = 40\%$.

6.1.3. Role of the Aspect Ratio L_m / H_m

The pattern aspect ratio L_m / H_m is a key design parameter of this approach. A square pattern (L_m / H_m)

would yield $e_x = e_y$, hence $C_{11} \equiv C_{22}$ and $E_x / E_y \equiv 1$: the composite would be in-plane isotropic for all volume fractions [18].

Conversely, an L_m / H_m ratio greater than 1 generates controlled anisotropy. The higher this ratio, the earlier the anisotropy appears (at low V_f) and the more pronounced it is at high volume fractions. For the 220×105 mm brick, the choice of $N \times M = 11 \times 10$ results in an L_m / H_m ratio of $20/10.5 = 1.905$; this ratio, rather than the fibre modulus, is the primary cause of the observed anisotropy.

6.1.4. Implications for design

The ratio E_x / E_y is the selection criterion for the practical zone.

- At $V_f = 7\%$: $E_x / E_y = 0.989$; the behaviour is virtually isotropic, and the brick can be laid in any orientation;
- At $V_f = 20\%$: $E_x / E_y = 0.882$; the anisotropy is moderate (-12% for E_x) and still compatible with simplified design methods;
- Above $V_f = 25\%$: $E_x / E_y < 0.80$. The anisotropy requires explicit consideration of the laying direction in structural calculations, which is incompatible with vernacular construction practices [37,38] (local artisanal practices [12,14]).

This analysis confirms that the choice of pattern geometry (L_m / H_m) is a more decisive factor regarding anisotropy than the choice of fibre.

6.2. The gain in the Transverse Young's Modulus

The transverse Young's modulus E_y increases quasi-linearly from 30.47 MPa at $V_f = 1\%$ to 40.8 MPa at $V_f = 15\%$ (see fig 3, component C_{22}), representing gains of $+1.6\%$ to $+36.0\%$ compared to pure clayey sand (47). This linear increase is consistent with Eshelby's theory of dilute inclusions [45], which predicts:

$$E_y^{\text{hom}} \approx E_2 (1 + \alpha V_f), V_f \ll 1 \quad (57)$$

where α is a concentration factor depending on the contrast E_1 / E_2 and the shape of the inclusion.

For the dilute regime, characterised by volume fractions ranging from 1% to 15%, the gain achieved is mechanical in nature. The gain of $+1.6\%$ to $+36\%$ is attributable to the intrinsic stiffness of the kenaf bundles. This result is experimentally reproducible: Millogo et al. [12] measured comparable gains in compressive strength for compressed earth blocks (CEBs) from Burkina Faso, and Ouedraogo et al. [13] obtained analogous results with adobe bricks reinforced with fonio straw. This gain can be directly applied in civil engineering, using volume fractions achievable through manual mixing. E_y becomes the sole governing parameter for axial loads. E_y reaches 275.6

MPa at $V_f = 40\%$ (see Figure 3, component C_{22}), representing an increase of $+818.8\%$ compared to pure clayey sand. This non-linear amplification stems from geometric confinement: as r_{eq} approaches $H_m / 2$, the residual clayey sand layer between adjacent bundles narrows, significantly stiffening the y -direction [18,19]. However, this gain is essentially geometric in origin and, beyond $V_f = 30\%$ is physically unattainable via mixing, thus representing a theoretical limit of the model.

6.3. The Most informative Modulus for Civil Engineering

The shear modulus G_{xy} is the modulus most sensitive to fibre content, showing a ratio of over 40.4 between $V_f = 1\%$ where $G_{xy} = 46.5$ MPa and $V_f = 40\%$ where $G_{xy} = 1,879.0$ MPa (see Figure 6). At $V_f = 5\%$, G_{xy} reaches 233.8 MPa, representing an increase of more than twenty-fold compared to pure clayey sand, for which $G_2 \approx E_2 / [2(1 + \nu_2)] \approx 11.1$ MPa. This high sensitivity is characteristic of composites with high stiffness contrast ($E_1 / E_2 \approx 1.667$) and single circular inclusions [22,27].

The increase in the shear modulus G_{xy} is quasi-linear across the entire range of $[1; 40\%]$, unlike the transverse Young's modulus E_y , which exhibits continuously accelerating growth, with a marked change in slope beyond $V_f \approx 30\%$. This behavior is explained by the fact that $G_{xy} = C_{33}$: the shear component of the tensor proves to be virtually insensitive to the intrinsic stiffness of the fibre and depends primarily on the shear modulus of the sandy matrix and the volume fraction of the reinforcement, rather than on the nature of the reinforcement itself. For bricks subjected to seismic loads or lateral forces such as wind and earth pressure, the shear modulus G_{xy} rather than the transverse Young's modulus E_y is the decisive design parameter [40,41]. This finding highlights the value of plant-fiber reinforcement for structures located in seismic or wind-prone areas, provided a moderate to high dosage ($V_f \geq 15\%$) is used: the gain in G_{xy} remains marginal below this threshold ($+1.6\%$ at $V_f = 1\%$, compared to $+22\%$ at $V_f = 1\%$ and $+80\%$ at $V_f = 40\%$) [9,13]. Furthermore, the fact that G_{xy} is insensitive to the type of fiber allows for the selection of locally most abundant resources without compromising shear performance.

6.4. Quasi-constant of the Component C_{12} : Original Result

The analysis of Figure 3 shows that the C_{22} component of the homogenised stiffness tensor increases from 34.70 MPa (at $V_f = 1\%$) to 279.90 MPa (at $V_f = 40\%$), a

factor of approximately 8.07, whereas the C_{11} component increases by a factor of more than 1.75. Both diagonal components thus increase steadily with the volume fraction V_f , but in an asymmetric manner. This increasing asymmetry, observed at volume fractions above 25%, is a direct consequence of the rectangular shape of the unit cell. The dimension $H_m = 10.5 \text{ mm}$ is smaller than $L_m = 20 \text{ mm}$, such that the inclusion saturates the y -direction first, leading to greater stiffening of the C_{22} component. A square unit cell ($L_m = H_m$) would yield $C_{11} \equiv C_{22}$ and thus transversely isotropic behaviour for all volume fractions [18]. Unlike the other components, the C_{12} component remains nearly constant across the entire range of volume fractions. More precisely, C_{12} increases slowly from 12.12 MPa at $V_f = 1\%$ to 16.13 MPa at $V_f = 40\%$, representing an increase of more than a factor of 1.3. The ratio C_{12}/C_{22} drops sharply from 0.349 to 0.058, physically reflecting the fact that Poisson coupling becomes increasingly negligible relative to directional stiffness as the composite becomes more concentrated. This behaviour promotes the stability of the brick under load, as it limits the transverse deformations that cause longitudinal cracking [40]. This novel result demonstrates that the tension/lateral contraction coupling is governed exclusively by the matrix, independently of the fibre volume fraction. It is consistent with the theoretical prediction that the off-diagonal terms of C^{hom} converge to the matrix values in the dilute limit [18,29]. The present study extends this result well beyond the dilute regime. This result has a direct implication for modelling. The C_{12} component can be estimated using the clayey sand matrix alone ($C_{12} \approx E_2 \nu_2 / (1 - \nu_2^2) \approx 11.97 \text{ MPa}$) with an error of less than 30% regardless of the volume fraction, which considerably simplifies model calibration in the absence of experimental measurements.

Recommended practical range

Based on the numerical results, the range $V_f \in [7\%; 20\%]$ offers the best compromise for civil engineering applications, a choice justified by:

- The mechanical gain of +13% to +58% in the transverse Young's modulus E_y is consistent with experimental measurements by Millogo et al. [12] on kenaf/clayey sand compressed earth blocks (CEBs) from Burkina Faso, and with those for sisal-reinforced adobe [6]. This gain is achievable with a dosage of 1% to 3% by mass (assuming $\rho_{kenaf} \approx 1.400 \text{ kg/m}^3$), a range compatible with manual mixing [13].

- The condition $E_x/E_y > 0.88$ ensures quasi-isotropic behaviour, which simplifies structural sizing and avoids directional strength characteristics that would complicate installation. Beyond 20%, the increasing anisotropy ($E_x/E_y < 0.88$) would require the installation direction to be explicitly accounted for in structural

calculations a requirement incompatible with vernacular construction practices.

- Below 20%, Von Mises stress concentrations at the kenaf/clayey sand interface remain moderate. The risk of premature delamination and micro-cracking is low [18,22]. Beyond this level, σ_{VM} concentrations at the inclusion poles increase sharply, a precursor to interfacial damage.

- A fibre content of 7% to 20% by volume corresponds to approximately 1% to 3% by mass a range confirmed to be compatible with manual mixing and good dispersion of the fibre bundles [4,13]. Ouedraogo et al. [4] achieved the best mechanical performance at 0.2%–0.4% by mass, either $\approx 3\%$ – 6% by volume, thereby validating the order of magnitude of the recommended range.

7. Conclusion

This work presented an original periodic asymptotic homogenisation (PAH) method applied to the clayey sand/kenaf composite, based on an area-conservation equivalence between N_f elementary fibres of radius r_f and a single equivalent bundle of radius r_{eq} such that $r_{eq} = r_f \sqrt{N_f}$. This construction inspired by the periodic cell of Pinho-da-Cruz et al. [18] and motivated by the demonstrated equivalence between periodic and random microstructures [19,20] constitutes the study's central conceptual contribution. The relationship $N_f = K \cdot V_f$ with $K = 57.824$ makes it possible to set the volume fraction a priori based solely on the integer N_f , without remeshing. Agreement between the geometric fraction and the fraction based on actual areas (T3) is verified to within $\pm 0.15\%$ across the 15 volume fractions. The Gmsh mesh of the entire $220 \times 105 \text{ mm}$ brick, comprising 90,000 T3 triangular elements, ensures node-to-node matching at opposite edges and thus exact periodic boundary conditions [22]. This choice eliminates the interpolation error systematically introduced by projection-based matching methods.

Despite a factor exceeding 8,1 in the C_{22} component of the homogenised stiffness tensor, the C_{12} component remains virtually constant across the entire range of volume fractions. This result demonstrates that the tension/lateral contraction coupling is an exclusive signature of the clayey sand matrix, independent of the fibre dosage and type [18,29]. In contrast, the shear modulus G_{xy} exhibits one of the least pronounced increases within the homogenised tensor, with a ratio of only 1.78 between $V_f = 1\%$ ($G_{xy} = 11.29 \text{ MPa}$) and $V_f = 40\%$ ($G_{xy} = 20.05 \text{ MPa}$) compared to a ratio of 9.37 for E_y over the same range. Thus, it is E_y , rather than G_{xy} , that proves to be the most sensitive to fiber content. The transition at $V_f \approx 15\%$ separates a quasi-isotropic dilute regime if $E_x/E_y > 0.94$ from an anisotropic

concentrated regime. The range $7\% \leq V_f \leq 20\%$ offers the best compromise, with stiffness gains ranging from +13% to +58% in the transverse modulus E_y .

The results are obtained within the framework of linear elasticity; ductility and toughness cannot be directly determined using the asymptotic homogenisation method. The mechanical properties of the matrix ($E_m = 30$ MPa, $\nu_m = 0.35$) are reference values from the literature [40,41] rather than values measured on the soil from the Ouatchi Plateau; a sensitivity analysis regarding $E_m \in [10;50]$ MPa and a local measurement are recommended. All Voigt–Reuss bounds are satisfied across the 15 fractions [48], confirming the thermodynamic consistency of the calculations. Extending the work to other fibres, geometrically optimising the pattern, and experimentally validating the approach using bricks manufactured in Togo are the priorities for the next stage of this research. We propose resuming our 3D work at a later stage.

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