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D- and G- Optimal Axial Slope Designs for Four Ingredient Mixture

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Abstract This paper aims at investigating and comparing the D- and G-optimal criteria for non-pure blends slope designs. The study used a parameter subsystem of interest based on the second-degree Kronecker model to obtain the H-invariant information matrices for both Equally Weighted Simplex Centroid Axial Design and Un-equally Weighted Simplex Centroid Axial Design. The D- and G- optimal values worked out revealed that the centroid achieved the best D- and G-optimality values and that the best D-efficient and G-efficient design points were η_1 with 105.71% and η_4 99.76% respectively. The latter design was more D-efficient while former design was more G-efficient.

Keywords: *axial design, subsystem, H-invariant, centroid, D-optimal, G-optimal, efficiency*

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1. Introduction

A mixture experiment with $q$components according to [6] satisfy the condition $\sum_{i=1}^{q} t_i = 1, 0 \leq t_i \leq 1, \quad i = 1, 2, \ldots, q$, where q is the number of components in the experiments. The aim of any experimenter is to obtain levels of the ingredients that optimize the expected response. In so doing, some optimality criterions chosen either minimize the variance or maximize information of the information matrix of the model adopted. [10] studied the optimal slope designs for second order Kronecker model mixture experiments for $q = 3$.

The researcher [8], introduced the q-component Simplex Centroid Design as designs where the $2^q - 1$ points are located at the boundaries, the vertices and the Centre of the $q - 1$ dimension simplex.

The q-component Simplex Centroid Design involves $2^q - 1 = q + \binom{q}{2} + \ldots + \binom{q}{r} + \ldots + 1$ distinct design points in total. There are q pure components $(1,0,0,\ldots,0)$ the $\binom{q}{2}$ permutations of binary mixtures $\left(\frac{1}{2},\frac{1}{2},0,0,\ldots,0\right)$, the $\binom{q}{3}$ permutations of ternary mixture $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0\ldots,0\right)$ and the $\binom{q}{4}$ permutation of quaternary mixture $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},0,0,\ldots,0\right)$ and so on up to mixtures involving $\left(\frac{1}{q},\frac{1}{q},\frac{1}{q},\ldots,\frac{1}{q}\right)$ of equal proportion of all q components or q-nary mixtures.

An Axial design is a design that consists mainly of complete or q component blends where most of the points are located inside the simplex. Axial designs are recommended to be used when component effects are to be measured, and in screening experiments. The simplest form of the axial design is one whose points are equidistant from the Centroid $\left(\frac{1}{q},\frac{1}{q},\frac{1}{q},\ldots,\frac{1}{q}\right)$, and has q axes [1].

2. Methodology

The study adopted an inscribed tetrahedral Simplex Centroid design also referred to as an axial design, for the fact that all the corresponding points in each face vertex and line of the design are equidistant from the original Simplex Centroid. This study chose to let the vertices of the simplex to be a distance h from the main vertices or $1 - h$ from the base the axis of the original Simplex. This original simplex is a regular tetrahedron of vertices (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1). This value h also denoted by Δ, has a maximum value given by $\frac{q-1}{q}$, according to [1]. The design suggested by [3] was developed in the following way

The four-vertex design points η_1 generated as

$$\left(1-\mathrm{h},\frac{\mathrm{h}}{3},\frac{\mathrm{h}}{3},\frac{\mathrm{h}}{3}\right),\left(\frac{\mathrm{h}}{3},1-\mathrm{h},\frac{\mathrm{h}}{3},\frac{\mathrm{h}}{3}\right),$$

$$\left(\frac{\mathrm{h}}{3},\frac{\mathrm{h}}{3},1-\mathrm{h},\frac{\mathrm{h}}{3}\right),\left(\frac{\mathrm{h}}{3},\frac{\mathrm{h}}{3},\frac{\mathrm{h}}{3},1-\mathrm{h}\right)$$

has a corresponding moment matrix $M(\eta_1)$.

The six edge mid points forming the design η_2 whose Moment Matrix is $M(\eta_2)$ were given by

$$\left(\frac{3-2h}{6},\frac{3-2h}{6},\frac{h}{3},\frac{h}{3}\right),\left(\frac{3-2h}{6},\frac{h}{3},\frac{3-2h}{6},\frac{h}{3}\right),$$

$$\left(\frac{h}{3},\frac{3-2h}{6},\frac{3-2h}{6},\frac{h}{3}\right),\left(\frac{h}{3},\frac{3-2h}{6},\frac{h}{3},\frac{3-2h}{6}\right),$$

$$\left(\frac{h}{3},\frac{h}{3},\frac{3-2h}{6},\frac{3-2h}{6}\right),\left(\frac{3-2h}{6},\frac{h}{3},\frac{h}{3},\frac{3-2h}{6}\right).$$

The four points on the faces of the inscribed tetrahedron forming the design η_3, whose moment matrix is $M(\eta_3)$ were the mixture blends generated as;

$$\left(\frac{3-h}{9},\frac{3-h}{9},\frac{3-h}{9},\frac{h}{3}\right),\left(\frac{3-h}{9},\frac{3-h}{9},\frac{h}{3},\frac{3-h}{9}\right),$$

$$\left(\frac{3-h}{9},\frac{h}{3},\frac{3-h}{9},\frac{3-h}{9}\right),\left(\frac{h}{3},\frac{3-h}{9},\frac{3-h}{9},\frac{3-h}{9}\right).$$

The fifteenth point, which forms the design η_4 is the Centre of the simplex design referred to as the Centroid $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$, whose moment matrix is $M(\eta_4)$.

The value $h = 0.3$ arbitrarily chosen, developed the fifteen-point design t' given as

$$\frac{1}{10}\begin{bmatrix} 7 & 1 & 1 & 1 & 4 & 4 & 4 & 1 & 1 & 1 & 3 & 3 & 3 & 1 & 2.5 \\ 1 & 7 & 1 & 1 & 4 & 1 & 1 & 4 & 1 & 4 & 3 & 3 & 1 & 3 & 2.5 \\ 1 & 1 & 7 & 1 & 1 & 4 & 1 & 1 & 4 & 4 & 3 & 1 & 3 & 3 & 2.5 \\ 1 & 1 & 1 & 7 & 1 & 1 & 4 & 4 & 4 & 1 & 1 & 3 & 3 & 3 & 2.5 \end{bmatrix}. \quad (1)$$

The Kronecker models are better models because they have increased symmetry resulting from the repletion of cross product terms, which result to larger moment matrices. The models are less susceptible to ill conditioning, which results to highly correlated parameters and large standard errors according to [4].

The polynomial function related to the K-model is

$$E[Y_i] = \sum_{i=1}^{q}\sum_{j=1}^{q} t_i t_j \theta_{ij} = (t \otimes t)' \theta. \quad (2)$$

The moment matrices obtained at each design point by the summation of Kronecker product shown in (3) are $q^2 \times q^2$ matrices.

$$M(\eta_i) = \int (t \otimes t)(t \otimes t)' d\tau. \quad (3)$$

The improved information matrices are the slope matrices which are obtained by utilizing the equation $D_c = HCH'$, which is the improved information matrix of the slope obtained from the $C = C_k(M)$ and H, the derivative of the elements of the design matrix $M(\tau)$ that is $H = \frac{dM(\tau)}{d\tau}$, where $M(\tau)$ is given by (4)

$$M(\tau) = t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4. \quad (4)$$

The general H derivative matrix of $M(\tau)$ given as (5)

$$H = \begin{bmatrix} 2t_1 & 0 & 0 & 0 & t_2 & t_3 & t_4 & 0 & 0 & 0 \\ 0 & 2t_2 & 0 & 0 & t_1 & 0 & 0 & t_3 & t_4 & 0 \\ 0 & 0 & 2t_3 & 0 & 0 & t_1 & 0 & t_2 & 0 & t_4 \\ 0 & 0 & 0 & 2t_4 & 0 & 0 & t_1 & 0 & t_2 & t_3 \end{bmatrix} (5)$$

For an arbitrary subset $\mathcal{H}$ of $s \times s$matrices, we define a symmetric $s \times s$matrix C to be $\mathcal{H}$ invariant if $C = HCH'$, for all $H \in \mathcal{H}$. The set of all $\mathcal{H}$ invariant symmetric $s \times s$ matrices are expressed as $sym(s\mathcal{H})$ as put by [5].

The optimality tests are usually done in order to locate the optimum values of a design according to each criterion, and are compared in order to obtain the design with the best characteristics. The *D, and G-Optimal values* for two designs namely, Equally Weighted Simplex Centroid Axial Design (EWSCAD) and Unequally Weighted Simplex Centroid Axial Design (UWSCAD) were calculated and compared in this study.

The designs obtaining the maximum information for the maximal parameter subsystem of interest $K'\theta$ are those that obtain maximum values through application of D and I optimality criterions, which satisfy the Kiefer-Wolfowitz equivalence theorem. The second-degree Kronecker full and subsystem of interest models for the four mixture components suggested by [2], are given in (6) and (7) respectively.

$$E(Y_i) = f(t)'\theta = \sum_{i=1}^{4} \theta_{ii}\, t_i^2 + \sum_{i,j}^{4} (\theta_{ij} + \theta_{ji}) t_i t_j. \quad (6)$$

$$E(Y_{i*}) = f(t^*)'\theta = \sum_{i=1}^{4} \theta_{ii}\, t_i^2 + \frac{1}{2}\sum_{i,j}^{4} (\theta_{ij} + \theta_{ji}) t_i t_j. \quad (7)$$

The experimenter uses the subsystem of interest with $\binom{q+1}{2}$ parameters, for three reasons; one, it is less expensive, two, there are no repeated terms that make the moment matrix rank deficient causing inefficiency in estimating the terms and three, it has the same characteristics as the full model.

The statistical package R-Gui version 4.0.0 was used to process the D- and G-optimal values.

2.1. Coefficient, Moment and Information Matrices

In this section coefficient, moment and information matrices K, M and C respectively were derived for the two designs.

2.1.1. Coefficient Matrix (K)

Due to repletion of terms in (6), experimenters find it more convenient to work with a subsystem of interest (7) whose coefficient matrix (8) less expensive. Therefore, they wish to study s components out of the

$k\ (s \le k)$. Equation (7) was achieved by integrating and averaging similar outcomes in (6). The linear parameter subsystem of interest $k'\theta$ for some $k \times s$ matrix K referred to as the coefficient matrix of the parameter subsystem $K'\theta$ develops the coefficients (8) as a result. It is estimable when there exists an unbiased linear estimator for θ. L is the left inverse of K such that $L = (K^{'}K)^{-1}K^{'}$. The squares and the cross products of the components of t for the subsystem of interest presented in Lexicographic order was given by (7).

$$\left(K'\theta\right)^{'} = \begin{bmatrix} \theta_{11}, \theta_{22}, \theta_{33}, \theta_{44}, \dfrac{\theta_{12}+\theta_{21}}{2}, \dfrac{\theta_{13}+\theta_{31}}{2}, \\ \dfrac{\theta_{14}+\theta_{41}}{2}, \dfrac{\theta_{23}+\theta_{32}}{2}, \dfrac{\theta_{24}+\theta_{42}}{2}, \dfrac{\theta_{34}+\theta_{43}}{2} \end{bmatrix}. \quad (8)$$

2.1.2. Moment Matrices ($M_{\eta e}$ *and* $M_{\eta u}$)

The general equivalence theorem of [5] defines the qualities and conditions for a competing moment matrix.

The moment matrices at the different design points obtained by (9)

$$M(\eta_i) = \int (t \otimes t)(t \otimes t)' d\tau \quad (9)$$

The general $q^2 \times q^2$ moment matrix is the equation (10)

$$M(\eta) = \alpha_1 M(\eta_1) + \alpha_2 M(\eta_2) + \alpha_3 M(\eta_3) + \alpha_4 M(\eta_4) \quad (10)$$

The moment matrices shown by (16) and (17) are worked out by the equations (11) and (12) for the two designs respectively

$$M_{\eta e} = \frac{1}{4}M(\eta_1) + \frac{1}{4}M(\eta_2) + \frac{1}{4}M(\eta_3) + \frac{1}{4}M(\eta_4) \quad (11)$$

$$M_{\eta u} = \frac{4}{15}M(\eta_1) + \frac{6}{15}M(\eta_2) + \frac{4}{15}M(\eta_3) + \frac{1}{15}M(\eta_4) \quad (12)$$

The $\alpha_i{}'s$ in (13) were worked out using the general equation $\alpha_i = \dfrac{\binom{q}{i}}{2^q - 1}$, $i = 1, 2, 3, 4$ and $q = 4$.

$$\begin{bmatrix}
\mu_4 & \mu_{31} & \mu_{31} & \mu_{31} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} \\
\mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} \\
\mu_{31} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} \\
\mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} \\
\mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} \\
\mu_{22} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_4 & \mu_{31} & \mu_{31} & \mu_{211} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{22} \\
\mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} \\
\mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{22} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{31} \\
\mu_{31} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} \\
\mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} \\
\mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{31} & \mu_{31} & \mu_4 & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{22} \\
\mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{31} \\
\mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} \\
\mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{31} \\
\mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} \\
\mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{31} & \mu_{31} & \mu_{31} & \mu_4
\end{bmatrix} \quad (13)$$

The matrix (13), is the general moment matrix for four components.

The values of the fourth order moments, which correspond with the moment matrix of the unequally weighted centroid design, are as represented in the set of equations (14).

$$\begin{aligned}
&\mu_4 = \frac{1}{2^q - 1}\sum_{j=1}^{15} t_{1j}^4 = 0.023054,\ \mu_{31} = \frac{1}{2^q - 1}\sum_{j=1}^{15} t_{1j}^3 t_{2j} = 0.006507 \\
&\mu_{22} = \frac{1}{2^q - 1}\sum_{j=1}^{15} t_{1j}^2 t_{2j}^2 = 0.004267,\ \mu_{211} = \frac{1}{2^q - 1}\sum_{j=1}^{15} t_{1j}^2 t_{2j} t_{3j} = 0.002767 \\
&\mu_{1111} = \frac{1}{2^q - 1}\sum_{j=1}^{15} t_1 t_2 t_3 t_4 = 0.001807.
\end{aligned} \quad (14)$$

Where t_1, t_2, t_3 and t_4 are given in (15)

$$\begin{aligned}
&\left(\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{4}{10}, \frac{4}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{10}, \frac{1}{4}\right) \\
&\left(\frac{1}{10}, \frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{1}{10}, \frac{4}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{10}, \frac{3}{10}, \frac{1}{4}\right) \\
&\left(\frac{1}{10}, \frac{1}{10}, \frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{4}{10}, \frac{3}{10}, \frac{1}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{4}\right) \\
&\left(\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{4}{10}, \frac{4}{10}, \frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{4}\right).
\end{aligned} \quad (15)$$

$$\frac{1}{10000}\begin{bmatrix} 207 & 60 & 60 & 60 & 60 & 41 & 30 & 30 & 60 & 30 & 41 & 30 & 60 & 30 & 30 & 41 \\ 60 & 41 & 30 & 30 & 41 & 60 & 30 & 30 & 30 & 30 & 30 & 22 & 30 & 30 & 22 & 30 \\ 60 & 30 & 41 & 30 & 30 & 30 & 30 & 22 & 41 & 30 & 60 & 30 & 30 & 22 & 30 & 30 \\ 60 & 30 & 30 & 41 & 30 & 30 & 22 & 30 & 30 & 22 & 30 & 30 & 41 & 30 & 30 & 60 \\ 60 & 41 & 30 & 30 & 41 & 60 & 30 & 30 & 30 & 30 & 30 & 22 & 30 & 30 & 22 & 30 \\ 41 & 60 & 30 & 30 & 60 & 207 & 60 & 60 & 30 & 60 & 41 & 30 & 30 & 60 & 30 & 41 \\ 30 & 30 & 30 & 22 & 30 & 60 & 41 & 30 & 30 & 41 & 60 & 30 & 22 & 30 & 30 & 30 \\ 30 & 30 & 22 & 30 & 30 & 60 & 30 & 41 & 22 & 30 & 30 & 30 & 30 & 41 & 30 & 60 \\ 60 & 30 & 41 & 30 & 30 & 30 & 30 & 22 & 41 & 30 & 60 & 30 & 30 & 22 & 30 & 30 \\ 30 & 30 & 30 & 22 & 30 & 60 & 41 & 30 & 30 & 41 & 60 & 30 & 22 & 30 & 30 & 30 \\ 41 & 30 & 60 & 30 & 30 & 41 & 60 & 30 & 60 & 60 & 207 & 60 & 30 & 30 & 60 & 41 \\ 30 & 22 & 30 & 30 & 22 & 30 & 30 & 30 & 30 & 30 & 60 & 41 & 30 & 30 & 41 & 60 \\ 60 & 30 & 30 & 41 & 30 & 30 & 22 & 30 & 30 & 22 & 30 & 30 & 41 & 30 & 30 & 60 \\ 30 & 30 & 22 & 30 & 30 & 60 & 30 & 41 & 22 & 30 & 30 & 30 & 30 & 41 & 30 & 60 \\ 30 & 22 & 30 & 30 & 22 & 30 & 30 & 30 & 30 & 30 & 60 & 41 & 30 & 30 & 41 & 60 \\ 41 & 30 & 30 & 60 & 30 & 41 & 30 & 60 & 30 & 30 & 41 & 60 & 60 & 60 & 60 & 207 \end{bmatrix} \tag{16}$$

$$\frac{1}{10000}\begin{bmatrix} 231 & 65 & 65 & 65 & 65 & 43 & 28 & 28 & 65 & 28 & 43 & 28 & 65 & 28 & 28 & 43 \\ 65 & 43 & 28 & 28 & 43 & 65 & 28 & 28 & 28 & 28 & 28 & 18 & 28 & 28 & 18 & 28 \\ 65 & 28 & 43 & 28 & 28 & 28 & 28 & 18 & 43 & 28 & 65 & 28 & 28 & 18 & 28 & 28 \\ 65 & 28 & 28 & 43 & 28 & 28 & 18 & 28 & 28 & 18 & 28 & 28 & 43 & 28 & 28 & 65 \\ 65 & 43 & 28 & 28 & 43 & 65 & 28 & 28 & 28 & 28 & 28 & 18 & 28 & 28 & 18 & 28 \\ 43 & 65 & 28 & 28 & 65 & 231 & 65 & 65 & 28 & 65 & 43 & 28 & 28 & 65 & 28 & 43 \\ 28 & 28 & 28 & 18 & 28 & 65 & 43 & 28 & 28 & 43 & 65 & 28 & 18 & 28 & 28 & 28 \\ 28 & 28 & 18 & 28 & 28 & 65 & 28 & 43 & 18 & 28 & 28 & 28 & 28 & 43 & 28 & 65 \\ 65 & 28 & 43 & 28 & 28 & 28 & 28 & 18 & 43 & 28 & 65 & 28 & 28 & 18 & 28 & 28 \\ 28 & 28 & 28 & 18 & 28 & 65 & 43 & 28 & 28 & 43 & 65 & 28 & 18 & 28 & 28 & 28 \\ 43 & 28 & 65 & 28 & 28 & 43 & 65 & 28 & 65 & 65 & 231 & 65 & 28 & 28 & 65 & 43 \\ 28 & 18 & 28 & 28 & 18 & 28 & 28 & 28 & 28 & 28 & 65 & 43 & 28 & 28 & 43 & 65 \\ 65 & 28 & 28 & 43 & 28 & 28 & 18 & 28 & 28 & 18 & 28 & 28 & 43 & 28 & 28 & 65 \\ 28 & 28 & 18 & 28 & 28 & 65 & 28 & 43 & 18 & 28 & 28 & 28 & 28 & 43 & 28 & 65 \\ 28 & 18 & 28 & 28 & 18 & 28 & 28 & 28 & 28 & 28 & 65 & 43 & 28 & 28 & 43 & 65 \\ 43 & 28 & 28 & 65 & 28 & 43 & 28 & 65 & 28 & 28 & 43 & 65 & 65 & 65 & 65 & 231 \end{bmatrix} \tag{17}$$

2.1.3. Information Matrices

The respective information matrices, $C_e = LM_{ne}L'$ and $C_u = LM_{nu}L'$, for the two designs are given by (18) and (19).

$$\frac{1}{10000}\begin{bmatrix} 207 & 41 & 41 & 41 & 120 & 120 & 120 & 60 & 60 & 60 \\ 41 & 207 & 41 & 41 & 120 & 60 & 60 & 120 & 120 & 60 \\ 41 & 41 & 207 & 41 & 60 & 120 & 60 & 120 & 60 & 120 \\ 41 & 41 & 41 & 207 & 60 & 60 & 120 & 60 & 120 & 120 \\ 120 & 120 & 60 & 60 & 164 & 120 & 120 & 120 & 120 & 88 \\ 120 & 60 & 120 & 60 & 120 & 164 & 120 & 120 & 88 & 120 \\ 120 & 60 & 60 & 120 & 120 & 120 & 164 & 88 & 120 & 120 \\ 60 & 120 & 120 & 60 & 120 & 120 & 88 & 164 & 120 & 120 \\ 60 & 120 & 60 & 120 & 120 & 88 & 120 & 120 & 164 & 120 \\ 60 & 60 & 120 & 120 & 88 & 120 & 120 & 120 & 120 & 164 \end{bmatrix} \tag{18}$$

$$\frac{1}{10000}\begin{bmatrix} 231 & 43 & 43 & 43 & 130 & 130 & 130 & 56 & 56 & 56 \\ 43 & 231 & 43 & 43 & 130 & 56 & 56 & 130 & 130 & 56 \\ 43 & 43 & 231 & 43 & 56 & 130 & 56 & 130 & 56 & 130 \\ 43 & 43 & 43 & 231 & 56 & 56 & 130 & 56 & 130 & 130 \\ 130 & 130 & 56 & 56 & 172 & 112 & 112 & 112 & 112 & 72 \\ 130 & 56 & 130 & 56 & 112 & 172 & 112 & 112 & 72 & 112 \\ 130 & 56 & 56 & 130 & 112 & 112 & 172 & 72 & 112 & 112 \\ 56 & 130 & 130 & 56 & 112 & 112 & 72 & 172 & 112 & 112 \\ 56 & 130 & 56 & 130 & 112 & 72 & 112 & 112 & 172 & 112 \\ 56 & 56 & 130 & 130 & 72 & 112 & 112 & 112 & 112 & 172 \end{bmatrix} \tag{19}$$

3. Results and Discussion

3.1. D-slope Optimal Values

This paper aims at obtaining slope information matrices at different points of the design in order to obtain the D- and G-optimal values for the two designs. The H- invariant slope matrices obtained from (6), for each design point were (20).

$$H_1 = \frac{1}{10}\begin{bmatrix} 14 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 7 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 7 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 7 & 0 & 1 & 1 \end{bmatrix}$$

$$H_2 = \frac{1}{10}\begin{bmatrix} 8 & 0 & 0 & 0 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 4 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 4 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 4 & 1 \end{bmatrix} \quad (20)$$

$$H_3 = \frac{1}{10}\begin{bmatrix} 6 & 0 & 0 & 0 & 3 & 1 & 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 3 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 7 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 3 & 3 \end{bmatrix}$$

$$H_4 = \frac{1}{10}\begin{bmatrix} 5 & 0 & 0 & 0 & 2.5 & 2.5 & 2.5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 2.5 & 0 & 0 & 2.5 & 2.5 & 0 \\ 0 & 0 & 5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 2.5 \\ 0 & 0 & 0 & 5 & 0 & 0 & 2.5 & 0 & 25 & 2.5 \end{bmatrix}.$$

Equation (21) below was used to obtain the determinants of the information matrices and hence the D-optimal values.

$$D = \left(\det\left(HCH'\right)\right)^{\frac{1}{p}} \quad (21)$$

P is the number of parameters under estimation, while C_e, C_u and H are as given in (18), (19) and (20) respectively. The following D-optimal values for both designs were calculated.

$$D_{1e} = 0.176753, D_{2e} = 0.1779787$$
$$D_{3e} = 0.1785106\ , D_{4e} = 0.1787608$$
$$D_{1u} = 0.186846, D_{2u} = 0.187484$$
$$D_{3u} = 0.1881068\ , D_{4u} = 0.1884468.$$

The efficiencies of the two designs were expressed using equation (22).

$$D_{eff} = \left\{\frac{|D_{iu}|}{|D_{ie}|}\right\}^{\frac{1}{10}} \times 100. \quad (22)$$

At each design point, the efficiencies were; 105.71%, 105.34%, 105.38%, 105.42%.

3.2. G-slope Optimal Values

The G optimality criterion also known as the global optimality criterion is a prediction criterion first introduced by Smith in 1918. It is a design that minimizes the worst case expected error in prediction. [6] provided the G optimality criterion definition as $MinMaxVar(\widehat{y_x})$ which is equivalent to;

$$MinMax\big(d(x\xi)\big) = MinMaxf^T(x)M^{-1}(\xi)f(x) \quad (23)$$

for a full parameter system and the subsystem of interest

$$MinMax\big(d^*(x\xi)\big) = MinMaxf^T(x)C^{-1}(\xi)f(x) \quad (24)$$

The inverse of the improved information matrix D_C^{-1} substituted for the inverse of the information matrix C^{-1} to obtain the G-slope optimal values. The exquation (25) gave the results of the G-slope optimal values.

$$MinMax\big(d^{**}(x\xi)\big) = MinMaxf'(x)D_{ciw}^{-1}f(x) \quad (25)$$

$w = e(equally\ weighted) or\ u(unequally\ weighted).$

[9] did define G-Optimal design as the design which minimizes the maximum variance of the estimated response function over the given design region. The minimum values of the maximum variances per design points were worked out and given as;

$$G_{1e} = 56.0556, G_{2e} = 22.1439$$
$$G_{3e} = 8.4233, G_{4e} = 3.9872$$

and

$$G_{1u} = 46.6038, G_{2u} = 20.8184$$
$$G_{3u} = 7.9170, G_{4u} = 3.9777.$$

The $G-$efficiencies worked out using the formula

$$G_{ef} = \frac{G_{iu}}{G_{ie}} \times 100 \quad (26)$$

Were, 83.14%, 94.01%, 93.99% and 99.76%.

4. Conclusion

This paper established the H-invariant matrices needed to derive the slope optimal values for the D- and G- optimal criteria. The study revealed that for the two designs, the centroid had the best D- and G- slope optimal values, that the best D-efficient design was η_1 with 105.71% and the most G-efficient was the centroid with 99.76%. UWSCAD was a more D-efficient design while EWSCAD was a more G-efficient design. To further this research, experimental data obtained using the adopted design would compare with these results, while other criteria would necessitate further findings using the same design.

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